DISPERSION RELATIONS IN THE HISTORICAL ASPECT

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Forty years ago N.N.Bogoliubov proved dispersion relations (DR) for πN elastic scattering amplitude [1]. Earlier DR have been formulated in papers of Gell-Mann, Goldberger, Thirring, Miyazawa, Nambu and Oehme [2-5].

But only after Bogoliubov's work it was recognized that DR (or more generally analyticity of an elastic scattering amplitude in s plane) are the fundamental properties of a scattering amplitude.

On the one hand DR allow us to check axioms that underlie quantum field theory (QFT),on the other hand they are a very effective tool for the investigation of elementary particle scattering.

DR show that axiomatic approach in QFT is very important not only as consistent and rigorous method in QFT, but also as a way to obtain nontrivial physical results.

In fact, analyticity of a scattering amplitude means:
1. the existence of correlations between low and high energies elementary particles physics;
2. the connections between the behavior of scattering amplitudes at physical and nonphysical s and t;
3. the connection between the behavior of the scattering amplitudes of particle and antiparticle, especially at high energies ;
4. the relations between the behavior of real and imaginary parts of the scattering amplitudes. Without analyticity real and imaginary parts of a scattering amplitude can be regarded as two independent functions.

Besides, analyticity in s plane is crucial for the derivation of many rigorous bounds on scattering amplitudes.

Why analyticity is such a nontrivial phenomena?

Why we can't simply admit it at the very beginning as we admit the continuity properties, the existence of the necessary number of derivatives and so on?

Very simple example can clarify this point.

Let's consider the function

$$F(s) = \int_{-\infty}^{+\infty} e^{isx} F(x) dx.$$

In accordance with Lehmann-Zimmermann-Symanzik reduction formulae a scattering amplitude has the similar form, only $F(x)$ is integrated over 4 dimensions. Let's suppose that the integral is meaningful at complex $s = s_1 + is_2$ as well. But $exp(-s_2 x)$ grows fast at $s_2 > 0$ and $x < 0$ (as well as at $s_2 < 0$ and $x > 0$). So we have no arguments for the existence of this integral and, consequently, for the direct extension of a scattering amplitude on such s.

It is necessary to stress that in spite of the fact that analyticity is a direct consequence of causality and other general principles of QFT the proof of analyticity is very complicated (only in the case $t = 0$ the proof can be sufficiently simplified, but even in this case the proof includes a few subtle points).

After the first papers the method of DR was developed in many directions.

Let's mentioned some of them:

1. The proof of DR for various amplitudes;
2. The generalization of mathematical methods;
3. The extension of the region of values of t, for which analyticity in s has been proved;
4. The derivation of the new relations which are rigorous consequence of analyticity (various sum rules, integral relations between real and imaginary parts of a scattering amplitude, logarithmic DR);
5. The derivations of many rigorous constrains on the behavior of scattering amplitudes, especially at high energies:

   i) Pomeranchuk type theorems,

   ii) Connections between the behavior of real and imaginary parts of the symmetric elastic scattering amplitude,

   iii) Upper and lower bounds on scattering amplitudes.

Unfortunately I have no time to speak about such important directions in theory of DR and topics related to them as:

1. DR for partial amplitudes,
2. DR and low energy physics of hadrons,
3. Dual models, Veneziano model and many others,
4. "Derivative" DR,
5. DR in non-local theories,
6. . . .

Now let's consider the directions mentioned above.
First of all I would like to stress that it is impossible to refer to all important papers in this field and I apologize that the list of references is not full.

DR for various processes were obtained in the papers of Bogoliubov and Shirkov [7], Logunov, Soloviev, Tavkhelidze, Bilenkij [8-11], Okubo, Polivanov, Chew, Goldberger, Low, Nambu, Jin and Todorov [12-16].

DR for decay processes were obtained by Goldberger and Treiman [17], Todorov and Khrustalev [18].

Very important step in this direction was done when Logunov obtained DR for inelastic (for instance, 2-3 particles) processes [19]. This work gave rise the study of inelastic processes and then inclusive processes in the frames of axiomatic field theory (Logunov, Mestvirishvili, Petrov and their collaborators, see, e.g. [20]).

Analytical properties of $3-3$ and $n-m$ scattering amplitudes were investigated in [21-23].

The mathematical methods were developed in papers of Bogoliubov, Vladimirov, Logunov, Medvedev, Polivanov [24-28]. The detailed proof of DR made in the book of Bogoliubov, Medvedev, Polivanov [29]. Others proofs of DR are due to Oehme [30], Symanzik [31], Bremermann, Oehme, Tailor [32], Lehmann [33]. In the last paper the analyticity in t variable has been also proved.

The further expansion of the validity of DR were obtained by Hepp [34] and especially by Bros, Epstein and Glaser [35], who proved that scattering amplitudes are analytical functions of s at all t, but the domain of analyticity is smaller. Such domain of analyticity implies the validity of so called "quasi-dispersion relations", which contain additional integral in complex s plane.

Various applications of scattering amplitude analytical properties can be illustrated by the simple example as follows.

Let $F(s)$ be a function analytical everywhere excluding the cut $(1, \infty)$. (In reality the scattering amplitude has also a left cut, but this fact doesn't change the conclusions given below). In this case analyticity implies for s outside the cut:

$$F(s) = \frac{1}{\pi} \int\limits_1^\infty \frac{ImF(s')ds'}{s' - s}. \tag{1}$$

We recall that $F(s - i\epsilon) = F^*(s + i\epsilon)$. Eq.(1) is valid if $F(s) \to 0$ at $|s| \to \infty$. In the real case in accordance with Froissart-Martin bound [36,37]

$$|F(s)| < \frac{\pi}{m_\pi^2} s ln^2 \frac{s}{s_0} \tag{2}$$

only the weaker condition is fulfilled, namely

$$F(s)/s^2 \to 0, \qquad s \to \infty. \tag{3}$$

This condition is also valid at complex s in view of Phragmen-Lindelöf theorem (see, e.g. [38]). Thus eq.(1) is valid for the function $[F(s) - F(0) - F'(0)s]/s^2$:

$$F(s) = F(0) + F'(0)s + \frac{s^2}{\pi} \int\limits_1^\infty \frac{ImF(s')ds'}{s'^2(s' - s)}, \tag{4}$$

The relations at $s > 1$ directly follow from (1) (or (4)) when $Ims \to 0$.

$$ReF(s) = \frac{1}{\pi} P \int\limits_1^\infty \frac{ImF(s')ds'}{s' - s}, \tag{1'}$$

$$ReF(s) = F(0) + F'(0)s + \frac{s^2}{\pi} P \int\limits_1^\infty \frac{ImF(s')ds'}{s'^2(s' - s)}, \tag{4'}$$

where "P" denotes the principal value of the integral.

To exclude unknown constants $F(0)$ and $F'(0)$ it is sufficient to consider proper combination of DR. After trivial calculations we come to the relation

$$\Phi(s_1, s_2, s_3) = \frac{1}{\pi} P \int_1^{\infty} \frac{ImF(s')ds'}{(s'-s_1)(s'-s_2)(s'-s_3)}, \tag{5}$$

where

$$\Phi(s_1, s_2, s_3) = \frac{ReF(s_1)}{(s_1-s_2)(s_1-s_3)} + \frac{ReF(s_2)}{(s_2-s_1)(s_2-s_3)} + \frac{ReF(s_3)}{(s_3-s_1)(s_3-s_2)}.$$

The dispersion integral in (5) contains the integration over very high (unattainable by experiment) energies. To get over this difficulty one has to take into account that according to optical theorem $ImF(s) > 0$ at $s > 1$, and thus the sign of non-observable part of the integral in (5) is positive. It gives the possibility to replace DR by finite energy inequalities which are valid independently on total cross section high energy behavior. Besides in accordance with bound (2) the unknown part of dispersion integral is of little importance if we consider DR at energies much lower than maximal energies attainable by experiment. In the most important case of πN elastic scattering such part has been rigorously estimated from above [39]. In principle it is possible to construct DR which contain integral convergent faster than in (5). To do this it is sufficient to consider DR for $\Phi(s_1, s_2, s_3) - \Phi(s_1, s_2, s_4)$. The proper choice of DR connects with the inaccuracy of experimental data. All tests of DR showed their correctness for hadron scattering amplitudes.

Let's point out that DR can be also used for the estimation of integrals over very high energies.

Very important consequence of analyticity is the existence of various sum rules. The first sum rules (finite energy sum rules) have been derived in paper of Logunov, Soloviev and Tavkhelidze [40].

Any sum rule can be obtained in such a way.

Let's consider $\int_C \rho(s)F(s)ds$, where $\rho(s)$ is an entire function, contour C consists of segments $(1+i0, s+i0), (s-i0, 1-i0)$ and circle with center in 0 and radius s, s is an arbitrary physical energy. Taking into account Cauchy theorem we obtain that

$$\int_1^s \rho(s)ImF(s)ds = -Im \int_{\bigcirc} \rho(s)F(s)ds \tag{6}$$

if $Im\rho(s) = 0$ at $s > 1$.
At high but finite energies a scattering amplitude is a smooth function. So we can with sufficient confidence continue it from physical s to complex s and calculate the integral over the circle. If $F(s)$ has a resonance between 1 and s, then left side of sum rules in main is determined by this resonance and sum rules give us the possibility to predict the parameters of resonances from the knowledge of high-energy behavior of the scattering amplitude and vice versa.

The choice $\rho(s) = s^n$ leads to finite energy sum rules.

The choice $\rho(s) = exp(\frac{-s}{s_0})$ leads to Borel sum rules (in this case we can put $s = \infty$). Borel sum rules were obtained in the paper of Shifman, Vainshtein and Zakharov [41] by another way. Finite energy and Borel sum rules were widely used in QCD (Krasnikov, Kuzmin, Matveev, Tavkhelidze et al. [42-46], Shifman, Vainshtein, Zakharov, Voloshin et al. [41,47], Bertlmann, Launer and Rafael [48], Narison [49] etc., see also review [50]).

The choice $\rho(s) = exp(\frac{-s}{s_0})^n$ leads to rapidly convergent sum rules (Vernov [51], Bertlmann, Launer and Rafael [48] (case n=2)). The properties of the latter sum rules have been investigated in papers of Fischer and Kolář [52]. General properties of Borel sum rules were considered in papers of Vernov and Mnatsakanova [53], Enkovszky and Struminsky [54].

Another possibility is to use relations between integrals of real and imaginary parts of the scattering amplitude.

Let us consider

$$\int\limits_{s_1}^{s_2} \rho(s) ReF(s) ds,$$

where s_1 and s_2 are arbitrary. Using DR (for simplicity eq.$(1')$) we come to the relation:

$$\int\limits_{s_1}^{s_2} \rho(s) ReF(s) ds = \frac{1}{\pi} \int\limits_{1}^{\infty} ImF(s') ds' P \int\limits_{s_1}^{s_2} \frac{\rho(s) ds}{s' - s}.$$

The attractive feature of this kind relations (especially important for theoretical investigations) is the substitution of the pole singularity in DR by more smooth (usually logarithmic) one. Relations of this kind were derived and investigated in papers of Vernov and Mnatsakanova [55] and Grunberg and Truong [56]. The properties of various integrals of scattering amplitudes were studied in papers of Fischer and Kolář [57], Lomsadze [58] and cetera.

DR were obtained not only for $F(s)$, but also for some functions of $F(s)$. The interesting example is DR for $lnF(s)$ - LDR (Jin and McDowell [59], Odorico [60], McClure and Jorna [61]). They can be derived as according to Jin and Martin result [62] $F(s)$ has no more than two zeroes. The last fact is the direct consequence of analyticity and positiveness of $ImF(s)$. Indeed, if eq.(1) is satisfied then it is evident that $F(s)$ has no zeroes at $s < 0$. If $s = s_1 + is_2$, then

$$ImF(s) = \frac{s_2}{\pi} \int\limits_{1}^{\infty} \frac{ImF(s') ds'}{(s' - s_1)^2 + s_2^2} \neq 0.$$

Thus $F(s)$ has no zeroes in the domain of analyticity. General case can be easily reduced to considered one. In fact, if $F(s)$ has two zeroes, for instance, at real points α and β, then $f(s) = \frac{F(s)}{(s-\alpha)(s-\beta)}$ satisfies eq.(1) and so has no zeroes.

Let's point out that owing to positiveness of $ImF(s)$ $\delta(s)$ - phase of $F(s)$ - satisfies the double inequality

$$0 \leq \delta(s) \leq \pi. \tag{7}$$

So the integral in LDR rapidly converges and the upper bound on unknown part of this integral can be easily obtained.

In fact $lnF(s)$ satisfies the condition

$$\frac{lnF(s)}{s} \to 0. \tag{8}$$

Thus only one subtraction in LDR is necessary. The condition (8) follows on the one hand from bound (2) and on the other hand from the lower bound on $F(s,0)$ derived by Jin and Martin [62]. The mentioned bound (refined by Wit [63]) is:

$$|F(s)| > Cs^{-2} \qquad \text{at} \qquad s \to \infty. \tag{9}$$

To prove this inequality it is sufficient to consider $\int_C F(s)s\,ds$, where contour C consists of real axis and semicircle in upper half-plane. Assuming that $F(s)s^2 \to 0$ at $s \to \infty$ and applying Cauchy theorem we come to the conclusion:

$$\int_1^\infty ImF(s)s\,ds = 0.$$

Thus $\sigma_{tot}(s) = 0$ at all s. Ineq.(9) is proved, but one comment is needed. In fact, ineq.(9) as well as other rigorous bounds is valid only at some sequence of $s_i, (s_i \to \infty$ at $i \to \infty)$. Under very reasonable (not rigorous) restrictions on the possible oscillations of an amplitude they can be extended for all asymptotic energies.

Let's stress that lower bounds are interesting in itself. They belong to rigorous restrictions on a scattering amplitude the existence of which crucially depends on the analyticity of $F(s,t)$ in s plane.

Other very important examples of such bounds are "Pomeranchuk type" theorems. In accordance with crossing symmetry :

$$F^+(s,t) = F^-(u,t) \qquad (10),$$

$F^+(s,t)$ is an elastic scattering amplitude of a particle, $F^-(u,t)$ is an elastic scattering amplitude of an antiparticle on the same target. At high energies $s \approx -u$. The behavior of an analytical function at $s \to \infty$ is tightly connected with its behavior at $s \to -\infty$. (We restrict ourselves only to asymptotic case).

So analyticity of the scattering amplitude gives rise very nontrivial relations between $F^+(s,t)$ and $F^-(s,t)$. The first result was obtained by Pomeranchuk [64]. He proved that if

$$\sigma^+_{tot}(s) \to C^+ \quad \text{and} \quad \sigma^-_{tot}(s) \to C^-,$$

then

$$C^+ = C^-$$

if only

$$\frac{1}{lns}\frac{ReF^+(s)}{ImF^+(s)} \to 0 \qquad at \qquad s \to \infty.$$

Generalizations of this result were obtained in many papers (Sugawara, Kanazawa [65], Weinberg [66], Meiman [67], Amati, Fierz, Glaser [68], Martin [69], Eden [70], Wit [71], Logunov, Mestvirishvili and Volkov [72], Cornille and Martin [73], Roy and Singh [74], Vernov and Mnatsakanova [75], Grunberg and Truong [56] Truong and Lam [76], Fischer and Kolář [57], Lomsadze et al. [77] and so on). There are detailed reviews of "Pomeranchuk type" theorems in papers of Logunov, Mestvirishvili and Khrustalev[38] and Fischer [78].

Different variants of "Pomeranchuk type" theorem can be uniquely obtained as follows.

For simplicity let's suppose that

$$F^{+}(-s+i\epsilon) = (F^{-})^{*}(s+i\epsilon). \tag{11}$$

Thus

$$F_a(-s+i\epsilon) = -F_a^{*}(s+i\epsilon), \qquad (11'),$$

where $F_a(s) \equiv F^{+}(s) - F^{-}(s)$. In reality these equalities are valid if we substitute s by $w = s - m^2 - M^2$ (m and M are masses of scattered particles) and consider $F(w)$. Evidently difference between s and w is negligible at $s \to \infty$. Let's assume that

$$\mid F_a(s) \mid < s(lns)^{\gamma}, \quad s \to \infty, \quad \gamma > 0. \tag{12}$$

Inequality (12) is also valid for all complex s if $|s| \to \infty$ in accordance with Phragmen-Lindelöf theorem:

$$\mid F_a(se^{i\varphi}) \mid < s(lns)^{\gamma}, \quad s \to \infty, \quad \gamma > 0. \tag{12'}$$

Let's consider $\int \frac{F_a(s')ds'}{s'^2}$ over the contour which consists of segments $(-s, -s_0)$, (s_0, s) and two semicircles in upper half plane with center in zero and radiuses s_0 and s. Owing to eq.(11), Cauchy theorem implies the relation:

$$\int_{s_0}^{s} \frac{\Delta\sigma(s')ds'}{s'} = -i \int_{0}^{\pi} \frac{F_a(se^{i\varphi})d\varphi}{se^{i\varphi}} + C(s_0). \tag{13}$$

where $\Delta\sigma(s') \equiv \frac{ImF_a(s')}{s'}$ and $C(s_0)$ denotes the integral over semicircle with radius s_0. From inequality (12′) the integral in right part of eq. (13) can be easily estimated from above and thus

$$\int_{s_0}^{s} \frac{\Delta\sigma(s')ds'}{s'} < \pi(lns)^{\gamma}. \tag{14}$$

Inequality (14) implies the existence of the sequence s_i ($s_i \to \infty$ at $i \to \infty$) such that

$$\Delta\sigma(s_i) < \gamma\pi(lns)^{\gamma-1}. \tag{15}$$

According to ineq.(2) $\gamma \leq 2$, thus in all cases $\Delta\sigma(s_i) < Clns$.

Evidently if $\gamma < 1$ $\Delta\sigma(s_i) \to 0$ at $s_i \to \infty$. By the same way it is easy to see that $\Delta\sigma(s_i) \to 0$ under the weaker condition $\frac{F_a(s)}{s\,lns} \to 0$ at $s \to \infty$ (Martin [69]). From ineq.(15) it follows that either $\Delta\sigma(s) = 0$ infinitely many times at high energies or $\Delta\sigma(s)$ at "almost all" high energies satisfies ineq.(15). Let's point out that only analyticity of $F(s)$ in upper half plane without semicircle of arbitrary radius s_0 is essential for the validity of the statement. In particular, it means that "Pomeranchuk type" theorems are valid for NN scattering. Ineq.(15) is a consequence of analyticity and additional condition (12).

The new important restrictions on $\Delta\sigma(s)$ can be obtained if one use in addition some consequence of unitarity and analyticity in t plane.

Results of this kind were obtained in the paper of Logunov, Mestvirishvili and Volkov [72] who used essentially the rigorous upper bound on $\frac{ReF^{\pm}(s)}{ImF^{\pm}(s)}$, which follows from unitarity and analyticity in t plane. According to Logunov, Mestvirishvili and Volkov results $\sigma^+_{tot}(s)$ and $\sigma^-_{tot}(s)$ at $s \to \infty$ only simultaneously either grow to infinity or tend to zero or are bounded by some constants from above and from below.

Besides in the most important case when total cross sections are unbounded

$$\frac{\sigma^+_{tot}(s)}{\sigma^-_{tot}(s)} \to 1. \tag{16}$$

The same result has been obtained by Eden [70] in case of $\sigma^{\pm}_{tot}(s) \sim (lns)^m$. In papers of Grunberg and Truong [56] it have been proved that for unbounded total cross sections $\frac{\sigma^+_{tot}(s)-\sigma^-_{tot}(s)}{\sigma^+_{tot}(s)+\sigma^-_{tot}(s)} \to 0$ in average. In fact if only $\sigma^{\pm}_{tot}(s)$ do not tend fast (as some power of s) to zero, then eq.(16) is valid in all cases when $\frac{ReF^+(s)}{ImF^+(s)}$ is bounded from above at $s \to \infty$. If

$$\left|\frac{ReF^+(s)}{ImF^+(s)}\right| \to \infty \qquad \text{then} \qquad \frac{ReF^+(s)}{ReF^-(s)} \to -1$$

(Vernov, Mnatsakanova [79]). So we come to the conclusion that $\frac{\sigma^+_{tot}(s)}{\sigma^-_{tot}(s)} \to 1$ for the wide class of possible (and in fact realistic) behavior of $\sigma^{\pm}_{tot}(s)$.

It is very interesting that analogous condition for differential cross sections proved by Logunov, Nguyen van Hieu, Todorov and Khrustalev [80], van Hove [81] and Meiman [82]:

$$\left|\frac{F^+(s,t)}{F^-(s,t)}\right| \to 1 \qquad \text{at} \qquad s \to \infty \tag{17}$$

is fulfilled for almost arbitrary behavior of $F^{\pm}(s,t)$. The only additional condition which is necessary relates with the possibility that $F^{\pm}(s,t)$ may have infinite number of zeroes. At $t = 0$ (precisely at t such that $t\,lns \to 0$) the necessary restriction of the number of the scattering amplitude zeroes are fulfilled an so for such t eq.(17) can be proved without any additional assumption (Kinoshita [83], Vernov [84]).

Ineq.(15) shows under what conditions $\Delta\sigma(s) \to 0$. At modern energies $\Delta\sigma(s)$ tends to zero sufficiently fast (as some power of s). So it is important to find under what conditions such behavior of $\Delta\sigma(s)$ is realized. The necessary conditions have been obtained in [38,67] and [85]. The result is that $\Delta\sigma(s)$ tends to zero as some power of s if $\left|\frac{ImF_a(s)}{ReF_a(s)}\right|$ is bounded from above at $s \to \infty$.

Other theorems for fast vanishing of $\Delta\sigma(s)$ were obtained in [38] and [57]. Let's point out that those "Pomeranchuk type" theorems which are obtained without using unitarity are, in fact, the consequences of the connection between the behavior of real and imaginary parts of antisymmetric amplitude.

The similar relations for symmetric amplitude, i.e. for sum of particle and antiparticle amplitudes, were first obtained in papers of Khuri and Kinoshita [86] and then were studied by many authors [38,56,59,77,87,88] etc.

DR exist not only for $F(s,t)$, but for all $\frac{d^nF(s,t)}{dt^n}$ as well, t is physical inside the domain of $F(s,t)$ analyticity. This important result has been proved by Martin [89].

Moreover the number of subtractions in DR for $F(s,t)$ and its derivatives is the same. Thus, if $F(s,t)$ satisfies eq.(1) (for simplicity we consider only this case) then

$$\frac{d^nF(s,t)}{dt^n} = \frac{1}{\pi}\int_1^\infty \frac{d^nA(s,t)}{dt^n}\frac{ds'}{s'-s}, \qquad (18),$$

where $A(s,t) \equiv ImF(s,t)$.
Eq.(18) is of decisive importance in an enlargement of $F(s,t)$ domain of analyticity in t plane from Lehmann to Martin ellipse. The most important step in such enlargement is the proof of the follows statement (Martin [89]):
Let $F(s_0,t)$ be an analytical function of t in some circle with center in zero and radius t_0. Then $F(s,t)$ is an analytical function of t in the same circle at arbitrary s. The proof follows directly from (18). For simplicity let's consider $s_0 < 1$ and s outside of the cut.

In accordance with the supposition made above the Tailor set $\sum\limits_0^\infty \frac{1}{n!}\left.\frac{d^nF(s_0,t)}{dt^n}\right|_{t=0}$ converges if $|t| \leq t_0$. But in view of (18)

$$\left|\frac{d^nF(s,t)}{dt^n}\right|_{t=0} < \mu(s)\left.\frac{d^nF(s_0,t)}{dt^n}\right|_{t=0}; \qquad \mu(s) = max\left|\frac{s'-s_0}{s'-s}\right| \quad \text{at} \quad 1 \leq s' \leq \infty. \qquad (19)$$

Indeed,

$$\left|\frac{d^nF(s,t)}{dt^n}\right|_{t=0} \leq \frac{1}{\pi}\int_1^\infty \left|\frac{s'-s_0}{s'-s}\right| \left.\frac{d^nA(s',t)}{dt^n}\right|_{t=0} \frac{ds'}{s'-s_0}.$$

Let's recall that owing to unitarity $\left.\frac{d^nA(s',t)}{dt^n}\right|_{t=0} > 0$, this statement follows directly from the expansion of $F(s,t)$ in Legendre series and the unitary condition for partial amplitudes.

It is evident that condition (19) is sufficient for the proof that $\sum\limits_0^\infty \frac{1}{n!}\left.\frac{d^nF(s,t)}{dt^n}\right|_{t=0}$ converges at arbitrary s.

Let's recall that if $t > 0$ belongs to Lehmann ellipse, then $t \sim \frac{1}{s}$ at $s \to \infty$.

The necessary analyticity of $F(s_0, t)$ has been proved by Bros, Epstein and Glaser [35]. It is easy to see that analyticity in the above mentioned circle implies the analyticity in the ellipse with focuses in points corresponded to the forward and to the backward scattering and the extreme right point t_0. Let's recall that in reality $t_0 = 4m_\pi^2$. (Sommer [90], Bessis and Glaser [91]).

It is of great importance that validity of DR at $t = 0$ leads to their validity inside Martin ellipse, what is more in accordance with the result of Jin and Martin [92] the number of subtractions (if even) remains the same.

Now let's turn to upper bounds on $F(s, t)$. The most important example of this kind of bounds is Froissart-Martin bound (2). It is remarkable that only extremely weak condition for possible $F(s, t_0)$ growth is necessary to obtain such bound.

Indeed, in paper of Logunov, Nguyen van Hieu and Todorov [93] it has been proved that under the usual Froissart arguments

$$|F(s, 0)| < C\frac{s^{n+1}}{t_0} \tag{20}$$

if

$$|F(s, t_0)| < exp(s^n). \tag{21}$$

Ineq. (20) implies that at least $n + 2$ subtractions are needed in DR for $F(s, 0)$, so at least $n + 3$ (if n is odd) - in DR for $F(s, t_0)$.

So we have obtained the new bound on $F(s, t_0)$:

$$\frac{F(s, t_0)}{s^{n+3}} \to 0. \tag{22}$$

The consequence of this condition is bound (2), in which $\frac{\pi}{m_\pi^2}$ is replaced by $\frac{4(n+2)\pi}{t_0}$. The final bound (2) derived by Łukashúk and Martin [37] follows from the fact that in view of the same arguments $\frac{F(s,t)}{s^2} \to 0$ at $s \to \infty, (t < t_0)$ and so in fact $n = -1$ in eq. (22).

Let's stress that only the convergence of $\int_1^\infty \frac{A(s,t)ds}{s^3}$ is necessary for the validity of the asymptotic bound (2). What is more the slightly stronger bound (s_0 is replaced by $(lns)^{\frac{3}{2}}$) is valid (see the second paper in [74]). For the existence of finite energy upper bounds it is necessary to have the additional restriction on this integral, namely:

$$\int_1^\infty \frac{A(s, t_0)ds}{s^3} < C, \tag{23}$$

where C is some known constant.

Let's recall that $A(s, t_0) > 0$ owing to unitarity. Thus

$$\int_{s_1}^{s_2} \frac{A(s, t_0)ds}{s^3} < C, \tag{23$'$}$$

where s_1 and s_2 are arbitrary points.

Ineq.(23′) implies the existence of upper bounds for various integrals of $A(s,0)$ over interval (s_1, s_2) or by the low of the mean the existence of the upper bound on $A(s,0)$ in some points inside (s_1, s_2).

The first bound of such kind for πN scattering was obtained by Ynduráin [94], who used the connection between the integral in (23) and D-wave scattering length in t channel. The various bounds of this kind were obtained in [95-97]. In paper [98] it has been proved that at high but finite energies bounds of this kind have the same functional form as Froissart–Martin bound.

Very interesting consequence of analyticity, unitarity and crossing symmetry is the existence of absolute (that is numerical) bounds for $\pi^0\pi^0$ elastic scattering amplitude. The first bound of this kind was derived by Martin [99]. The improved bounds were obtained in paper of Łukashúk and Martin [37] and then in papers [100-104]. Above mentioned bounds are bounds in nonphysical region, where $|s| < 4m_\pi^2$, $|t| < 4m_\pi^2$ and $|u| < 4m_\pi^2$. As it has been shown by A.Martin [99], such bounds in fact imply the restrictions on the physical amplitudes, namely, on the integrals of the scattering amplitude over all physical region.

At any physical energy the upper bound on $\pi^0\pi^0$ total cross section without unknown or phenomenological constant has been obtained by Mnatsakanova and Vernov [105]. This bound is the consequence of analyticity in s plane and the above mentioned absolute bounds.

In conclusion let's briefly consider the status of DR in gauge theories. Two problems arise when we try to extend the usual methods of proving DR on gauge theory, say QCD. The first problem is related with the fact that in covariant gauges the underlying space is not Hilbert one but indefinite metric space. The second is related with confinement. Nevertheless as it was proved in series of Oehme's papers [106] under some additional but very natural assumptions it is possible to prove that DR remain hold for gauge theories.

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