GRAVITATIONAL LENSES

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A brief description of the history of gravitational lenses is given. Discussed are the basic principles of the gravitational lens theory, in particular, the gravitational lens equation, multiple images, a magnification of images.

1. Introduction

The deflection of light is known to be described by equations of the general relativity theory. However, the bending of light rays discussed soon after the discovery of the classical mechanics. Newton noted Query 1: *"Do not bodies act upon light at a distance and is not this action strongest on the least distance?"* [1]. Thus, one can say, that Newton did, indeed, formulate the law about the bending of light in a gravitational field

$$\Theta = \frac{2GM}{c^2 R}. \tag{1}$$

To be more accurate, $\Theta \sim R^{-1}$, since the numerical coefficients did not define at that time because there were not common systems of units. Moreover, in Proposition XII (Problem VII) Newton considered a motion of a a test body in a hyperbola, it follows that acceleration is decreased as the unit divided by squares of the distance between a test body and the gravitational center [2]. Similar assertions were proved in the assumption that a body moves in a parabola (Proposition XIII, Problem VIII) and in an ellipse (Proposition X, Problem VIII) [2]. In Corollary 1 to Proposition XIII Newton asserted that a body attracted by an unverse-square centripetal force toward the gravitational center moves in a conic section having focus at the fixed point of attraction [2]. Thus, in spite of recently appearing Weinstock paper in which he tried to refute a well-known assertion that Newton proved that an inverse-square force acting on a body in a gravitational field requires that the body moves in the conic-section orbit (ellipse, hyperbola or parabola) [3], it is necessary to agree with Arnold's arguments and admit that the proof of the fact - that an orbit will be only a conic section - was basically formulated by Newton [4].

Since Newton calculated semi-axes during consideration a body motion in a hyperbola, so the angle between asymptotes was known to him. Thus, taking into a consideration the fact that very often Newton did not published his discoveries and presented the results in non-ordinary form, one can assert that equation (1) was known to Newton. Moreover, probably he knew the corresponding value of the bending angle of light near the Sun because all necessary constants had been known by the time of a publication Principia. Actually, Cassini calculated the distance between the Earth and the Sun in 1672, the period of revolution of the Earth around the Sun was calculated by Tikho Brahe (1546-1601) with accuracy up to 1 s [5]. Knowing the distance between the Sun and the Earth and the period of revolution the Earth one can calculate according to the third law of Kepler $GM_{\odot}$, because $T^2/a^3 = 4\pi^2/(GM_{\odot})$. The results of measuring the speed of light obtained by Roemer on the basis of changing the period of revolution of the satellite Io of Jupiter were published in 1677. So, as Bliokh and Minakov noted, the value of all necessary constants had been known by 1697 (the year of the Principia publication) [5]. Nevertheless, there is not the equation (1) in the Newton's scientific papers as well as the value of the bending of a light ray near the Sun is also absent, it may be because of its very small value - $0''.87$. As a historical irony we can note as Bliokh and Minakov did [5] that if one calculate the bending of a light ray near the Sun, using the deflection of light calculated by Roemer $c = 2 * 10^{10} cm/s$, then the angle of the bending would be equal to $\Theta = 1''.73$. The result almost coincides with a result obtained in the framework of the general relativity theory. In spite of that equation (1) was apparently known to Newton, for the first time the angle of the bending (1) was mentioned only in 1784.

At that time Henry Cavendish stimulated be the correspondence with his friend John Mitchell (who in 1783 a few years before Laplace considered a star with so high gravitational field that the light ray could not leave its surface) derived equation (1). Cavendish did not publish his calculations, nevertheless, these were written *"on the separate list of paper"* [6]. In 1801 a German astronomer Johann George von Soldner submitted to the Berlin astronomical annual book the paper about the bending of light in the gravitational field of a star. (The paper [7] was published in 1804). Therefore, the derived formula (1) was for the first time published by Soldner [7].

The fact that this remarkable Soldner's paper [7] was not known to the scientific world about 120 years (as well as a lack of reference to his name in the papers of mechanics and the theory of gravitation) make us to remind some facts of his life and achievements [8]. He was born on July 15, 1776 in a Bavarian village Georgenhof. He was the son of a peasant, studied one or two years in the elementary school in Feuchtwangen but he never had formal schooling. He was already sixteen when he got private instructions in Latin and French. He was twenty years old when his friends steered him to a scientist in Ansbach. That Soldner was a person with unusual talents which were further recognised and an official of the Prussian government in Ansbach suggested him to work as assistant of Bode in the Berlin Observatory. It was through publications in the widely read scientific journals particularly Bode's Astronomisches Jahrbuch (annual book), that his name became known. Besides the investigation of the bending of light by gravitated body Soldner considered the motion of stars in the Milky Way, analysed the path of the comet newly discovered by Piazzi. Remind, that this comet turned out to be the first asteroid. Soldner's article [9] about the motion of stars in the Milky Way was very interesting and devoted to an analysis of invisible bodies of Laplace [10], i.e. bodies with so strong gravitational field that light can not leave its surface. For the first time Laplace mentioned about these bodies in the first and the second editions of his Exposition du systeme du monde (1796, 1799) (but then the passage was omitted in the third (1808), fourth (1813) and fifth (1824) editions for reasons which Laplace did not specify).

One of very few, who responded to Laplace's idea was Soldner. For Soldner the mathematics of Laplace was unobjectionable, nevertheless, Soldner did not agree with his hypothesis about the constansy of the speed of light. According to Soldner's opinion the processes of light emission can be very variable and, thus, light corpuscles can be emitted with a relatively small velocity. So, in dependence on chemical and mechanical condition of its material, a star could even be very small and still trap the light emitted by it. Soldner considered a possibility of an existence of a large mass in the centre of a star system like the Milky Way. That the observable star systems exist could not be doubted since Kant, Lambert, and Herschel. But whether their motion was governed by a massive body in the centre of the Milky Way there was the question analysed by Soldner. He concluded that there are not a central body because the motion of stars around it would be too fast not to be noticed. Soldner concluded this having calculated the relative velocities of stars closer and farther from the center of a homogeneous distribution of stars, resembling a highly flattened spheroid. He supposed that all visible stars were so close to the center as they to be within the one-tenth of the radius of the Milky Way, but even at this case an angular speeds of stars could differ perceptible. As a result, he concluded that the hypothesis of the existence of a massive body (even in the form of Laplace's invisible stars) are not confirmed.

In 1804 the first Soldner paper on geodesy was published. There the Swedish project of measuring the length of a degree of an arc of the Earth and also his own proposal for a similar project in equatorial Africa were discussed. Undoubtedly, thanks to the impact of these papers he had been asked three times to accept the directorship of the Observatory of the University of Moscow. In that time his friends, who wanted to keep his talents for Germany secured for him the appointment in 1805 as director of Prussian land-surveying in the Ansbach district. Three years later Soldner was in the same capacity in the service of the Bavarian government. Nevertheless, his very rapid rise caused the envy of some and a year after his appointment to the Munich Academy he was recalled since he had been accused of plagiarism. The charge concerned the originality of a famous Soldner's paper on the reduction of observed azimuths published in 1813 in Bode's Jahrbuch. It should be noted that the priority of Soldner was defended by Delambre who asserted that Soldner's method was *"fortunate and new"*. As a result of a detailed inquiry Soldner was not only fully rehabilitated but his accuser, Felix Seyffer, member of the Munich Academy and director of its observatory, was removed from his post, and Soldner was appointed to this one. This appointment gave him an opportunity to plan a new observatory. By 1818 the observatory in Bogenhausen near Munich had been completed and equipped with the best instruments. These ones were made in the workshop of Fraunhofer, who later owing to Soldner's recommendation was nominated to the Munich Academy. In 1825 he was knighted by the King of Bavaria and the King of France, and also thanks to recommendation of the younger Herschel was elected a corresponding member of the Royal Astronomical Society in London. Meanwhile, Soldner was forced owing to a liver ailment to charge his assistant Johann Lamont to carry out the observational work, which had to do with the question of the motion of stars. Despite the fact, that these subject was the topic of Soldner's first publication he did not printed his papers in astronomy for a long time.

Only in 1811 he published a part of his private letter under the title *"On the Theory of Light, of Heat and about a Work from Integral Calculus"*, where he proposed to consider light to be a fourth state of material which is different from solid, liquid and vaporous state. Further, Soldner carried out a lot of useful investigations in the field of geodesy and astronomy. Thus, in Ostwald's Klassiker der Naturwissenschaften was pulished Soldner's paper about a method of computing triangles on a flattened spheroid such as the earth. The essence of his method is the calculation of the length of arcs with an error not greater than 1 cm over several kilometres instead of substitution ones by chords. Up to his death on 18 May, 1833 Soldner occupied the post of director of Bogenhausen observatory.

After his death Franz Gruithuisen, professor of astronomy at the University of Munich, but a weak in mathematical techniques was proposed on the post of director of this observatory. This candidacy caused a concern even to Shelling, president of the Munich Academy, who as a chief proponent of Naturphilosophie, was hardly a friend of exactness in the sciences. Thanks to the recommendations of Gauss and Bessel the successor of Soldner at the observatory was Lamont. He kept alive Soldner's renown as a champion of precision and new mathematical techniques both in astronomy and geodesy.

In spite of that Soldner was a well-known scientist his essay about the bending of light was almost forgotten. The significance of that Soldner's paper was clear to scientific community only after an appearance of Einstein's paper. In 1911 Einstein got within the framework of the special theory of relativity exactly the same value of the bending of light around the Sun as had already been calculated by Soldner (calculating with using equation (1)) [11]. At the beginning of XX century astronomers could already measure such bending. Actually, by the 20th years of XIX century through the work of Fraunhofer telescopes, which resolving power enabled to measure the calculated effect had been available. These instruments enabled to Bessel and Struve to measure for the first time stellar parallaxes which were mush smaller than 1". The group of astronomers from Berlin observatory called attention offered by the total eclipse of August 21, 1914 in the Russian Crimea. They actually had plans to test the predicted value. They have already been in the Crimea when the First World War began. The astronomers were captured by Russians and then they were exchanged for Russian citizens captured by Germans. So, according to Brecher's remark, fortunately for Einstein, this measurement was not made [12].

Only within the framework of the general relativity theory did Einstein [13,14] get the value twice as great as the one calculated in 1911

$$\Theta = \frac{4GM}{c^2 R}. \tag{2}$$

Soon after the ending of the First World War the observations of the bending of light were realised during the total eclipse of May 29, 1919 [15]. As noted in the publication of the results of the expedition, the main purpose of it was to choose one of three alternatives [15]:
1) the gravitational field of the Sun does not influence on bending of light;
2) the gravitational field of the Sun influences on bending of light according to the Newtonian law of gravity, so the deflection angle of a ray near the Sun is equal $0''.87$;
3) the gravitational field of the Sun influences on bending of light according to the general relativity, so the deflection angle of a ray near the Sun is equal $1''.74$.

In the paper [15] it was noted that for the deflection in framework of Newtonian gravitational had been calculated by Prof. A. Einstein using the relativity principle.

The observations were carried out at Sobral in northern Brasil and on the island of Principe off the western coast of Africa. The astronomers at Sobral measured the deflection of $1''.98 \pm 0''.12$, and the group on Principe did the one of $1''.61 \pm 0''.30$.

Moreover, at Sobral there were the plates which gave the deflection of $0''.93$. These results were rejected, since they had errors which significantly exceeded the random error. Thus, the results of the observations confirmed the Enstein's prediction. Nevertheless, the German Nobel winner P. Lenard noted [16], that it was in the paper of Soldner that equation (1) was asserted first. Lenard gave the part of Soldner's paper in his article and proved that the results were accorded with the prediction of the deflection angle about $0''.87$.

Nevertheless, today the deflection angle has been confirmed with an accuracy better than 1% at radio frequencies [17, 18]. It seems, that the first one who used the term "lens" in the context of the deflection of a light ray by gravity was Oliver Lodge, although he remarked that *"not permissible to say that the solar gravitational field acts like a lens for it has not a focal length"* [19].

Chwolson, a Petersburg's physicist, published a brief article [20] in which he noted that if one considered the deflection of a light ray from a background star by a star - lens there was a probability of an appearance of a secondary image of the background star, but the angle between the images would be too small to be resolved using optical telescopes. In case when an observer, a lens and a light source arrange in a straight line then a ring-shaped image will be appeared. According to note of the American astronomer Barnothy [21] these circles which are called now as *"Einstein rings"* should be named as *"Chwolson rings"*, and P. Schneider and others noted *"the biggest cat gets all the milk"* [22]. It is necessary to note, that the cause of the greater popularity of Einstein's paper is not only his greater renown in comparison with Chwolson but also the journal "Science" was more popular then Potsdam Astronomical Journal "Astronomische Nachrichten".

In 1936 Einstein published at the request of R.W. Mandl, a Czech electrical engineer, a brief article [23]. Einstein considered the deflection, due to a star, of the light from a background star, in particular, appearing a ring in case if an observer, a lens and a source perfectly aligned ("Einstein rings" appear). However, Einstein noted that, *"there is no great chance of observing this phenomenon".* Below, we will discuss Einstein's note. Besides, the article included calculating the magnification coefficient of the image of a source by a gravitational lens. It should be noted that both Chwolson and Einstein came to a conclusion that the phenomenon of gravitational lens could not be observed (as regards of observing a double star or "Einstein rings") in case when source and gravitational lens both are stars. Nevertheless, Fritz Zwicky concluded that the effect could be observed if a source is a nebula and gravitational lens is a galaxy.

Like Schneider and others [22], we give the quotation from paper of Zwicky [24]: *"Last year Dr. V.K.Zworykin (to whom the same idea had been suggested by Mr.Mandl) mentioned to me the possibility of an image formation through the action of gravitational field. As a consequence I made some calculations which show that extragalactic nebulae offer a much better chance that stars for the observation of gravitational lens effects."* This quotation may be the sample of a scientific paper style more than half century ago with references to other scientific ideas, not only references to published papers. It is to see that is not a single sample since we recall the following quotation from Einstein's paper [23]: *"Some time ago, R.W. Mandl paid me a visit and asked me to publish the results of a little calculation which I had made at his request. this note complies with his wish".* The quotation from Zwicky's paper shows that old Russian scientific school extensively influenced on a development of a world science, in particular, the influence of such brilliant representatives as V.K. Zworykin, whose technical discoveries were so great that it is possible to call him as the *"father"* of a television. There were obvious forty and two years later that Zworykin's remark and, of course, Zwicky's consideration was extremely fruitful. Really, when in 1979 Walsh and others detected the first gravitational lens when they observed the double quasar QSO 0957+16 A,B ($z \approx 1.4$, the angular distance between the images about $6''$) and galaxy is a gravitational lens ($z_d \approx .36$), one can say that the prediction of Zwicky was confirmed [25]. However, Zwicky's paper was forgotten and only Liebes [26] and Refsdal [27] once again considered the effect of gravitational lensing by a point body and investigated it in detail. In particular, Refsdal proved a possibility to use the geometrical optics approximation [27].

By the beginning of 1994 there had been discovered 13 objects, connected with gravitational lenses and 8 candidates for gravitational lens to be verified in the future, besides 5 radio arcs, which are also connected with gravitational lenses [28].

It should be noted that the discoveries of first microlens by MACHO collaboration [29] and EROS collaboration [30] were definitely the most important observational astronomical results in 1993. There is a popular discussion of the remarkable results in the paper [31].

Since from 1979 the number of papers devoted to gravitational lens are so great it is difficult to mention even the most significant ones.

Therefore we recommend the monograph by Bliokh and Minakov [5] (the unique book on gravitational lenses in Russian),
the monograph by Schneider, Ehlers and Falco [22] (the unique book on gravitational lenses in English),
the remarkable review by Refsdal and Surdej [28] for detailed studies of the subject.

2. Basics of Gravitational Lens. The point mass lens model

Let's consider the basic principles of the theory of gravitational lens on the example of a point gravitational lens. Suppose that light propagates not along a curve like a hyperbola but along the asymptotes of that curve. It is clear that the perfect gravitational lens is achromatic, however, really the different parts of the source can have different colours and, thereby, "chromatic" effects may arise. So, one can consider the ideal point gravitational lens.
The distance between the source and the observer is D_s,
the distance between the lens and the observer is D_d ,
the distance between the lens and the source is D_{ds}.
Draw a plane through the point where the gravitational lens is (the lens plane). This is a plane which is orthogonal to the line between the lens and the observer. Then one can draw a source plane being parallel to the lens plane from a source point. So, the equation of the gravitational lens is (in the assumption that a deflection angle is small) (see, for example, the book [22])

$$\boldsymbol{\eta} = D_s \boldsymbol{\xi}/D_d - D_{ds}\boldsymbol{\Theta}(\boldsymbol{\xi}), \tag{3}$$

where the vectors $\boldsymbol{\eta}, \boldsymbol{\xi}$ define the coordinates in the source plane and in the lens plane accordingly and the angle is defined from the equation

$$\boldsymbol{\Theta}(\boldsymbol{\xi}) = 4GM\boldsymbol{\xi}/c^2\xi^2. \tag{4}$$

Being equal the right part of equation (3) to zero, one get a case when a source, a lens and an observer arranged on a straight line ($\boldsymbol{\eta}= \mathbf{0}$). The given value $\boldsymbol{\eta} = \mathbf{0}$ corresponds to $\xi_0 = \sqrt{4GMD_dD_{ds}/(c^2D_s)}$ - the Einstein-Chwolson radius. One can calculate the Einstein-Chwolson angle - $\theta_0 = \xi_0/D_d$. We calculate the angle θ_0 for two typical cases. The first one is when both lens and source are stars, and the second one is when a source is a quasar and a gravitational lens is a galaxy. One consider that $D_s \gg D_d$ then

$$\theta_0 \approx 2'' \times 10^{-3} \left(\frac{GM}{M_\odot}\right)^{1/2} \left(\frac{kpc}{D_d}\right)^{-1/2}.$$

If a gravitational lens is one of the nearest galaxy with mass $M = 10^{12} M_\odot$, $D_d = 100 kpc$ is the distance of the lens from observer, then $\theta_0 \approx 200''$. If $1 kpc$ is the distance of the lens from an observer (the lens may be in our Galaxy) and the lens has a mass of the order the Sun's mass $M = M_\odot$, then we have $\theta_0 \approx 2'' \times 10^{-3}$.

Write the lens equation being normalize the angular variables with the Einstein-Chwolson angle by using the variables

$$\boldsymbol{x} = \boldsymbol{\xi}/\xi_0, \quad \boldsymbol{y} = D_s \boldsymbol{\eta}/(\xi_0 D_d),$$
$$\boldsymbol{\alpha} = \boldsymbol{\Theta} D_{ds} D_d/(D_s \xi_0) \tag{5}$$

one arrives at the following lens equation

$$\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{\alpha}(\boldsymbol{x}) \tag{6}$$

or

$$\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}/x^2. \tag{7}$$

Solving the equation about $\boldsymbol{x}$, one arrives at

$$\boldsymbol{x}^{\pm} = \boldsymbol{y}[1/2 \pm \sqrt{1/4 + 1/y^2}]. \tag{8}$$

Using the preceding equation, one can easily calculate the distance between the images.

$$x^+ = y\left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{y^2}}\right], \tag{9}$$
$$x^- = y\left[-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{y^2}}\right], \tag{10}$$
$$l = x^+ + x^- = 2y\sqrt{\frac{1}{4} + \frac{1}{y^2}}. \tag{11}$$

It is not difficult to understand why does the double image of a source appears while there is an inversely proportional dependence of the angle of the bending of a light ray on an impact parameter. The same reason are correct if there is a monotone decreasing function on an impact parameter (the function describes the deflection of a light ray). Therefore, the following discussion is correct for the case also.

One consider a plane $u_1 0 v_1$, where there are a source, a gravitational lens and an observer. Suppose that a gravitational lens is at the origin of coordinates, an observer is on the axis of abscissa ($u_1 > 0$), and the coordinates of a source are ($u_1 < 0, v_1 > 0$). Suppose that the source emits light rays in all directions then there is a light ray which arrives at any point on the half-line ($u_1 > 0, v_1 = 0$). That is the result of the dependence of the deflection angle on an impact parameter, namely

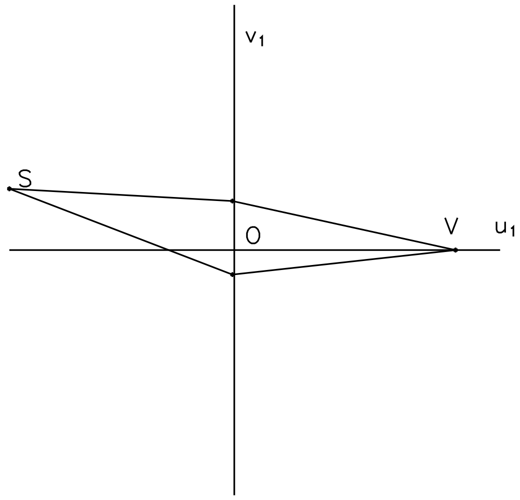

Figure 1: A point gravitational lens is at the origin of coordinates (0,0). The point S defines the source position, the point V does the observer position.

monotone decreasing of the deflection angle on an impact parameter $\Theta \sim p^{-1/2}$. Similarly, there is a direction of a light ray which arrives at any point on the half-line $(u_1 > 0, v_1 = 0)$ and the light ray lies partly on the half-plane $(v_1 < 0)$. So, if we consider a point source and a gravitational lens is a point body, then always there are two images except the degenerate case in which a source, a gravitational lens and an observer are on a line (we see the Einstein ring in the case). However, really anyway there is a some distance of the source from the line $v_1 = 0$, therefore there are two images of the source. If the distance of the source from the line $v_1 = 0$ is great enough then the second image is screened by gravitational lens in real astrophysical samples, so the point lens approximation is not acceptable for the samples.

If we consider the case $y \ll 1$, then $l \approx 2$, so the distance between images is about the Einstein radius. The circular sources and its images are represented for different distance of the source from the origins in Fig. 2 - 4. It is easy to understand why the circular source has so stretched images. Really, if we consider the source near the origin $y \ll 1$, then its images lie near the Einstein circle (x_+ is outside the circle, x_- is inside the circle). If we consider also the arc, which has a center at the origin, (ends of the arcs coincide with ends of the source diameter, which is orthogonal to the abscissa line), then there is the expansion of images in the tangential direction in about $1/y$ times.

Let us now consider the changes of a radial size of the source for the gravitational lens mapping. We suppose that the source center lies on the abscissa line (without loss of generality)

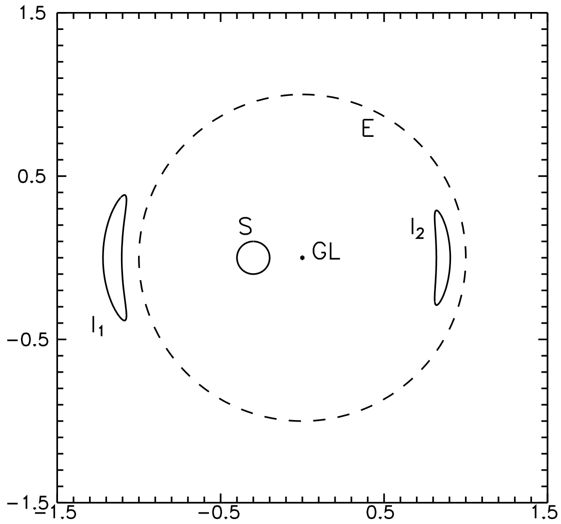

Figure 2: The circular source and its images which are distorted by point mass lens. The unit circle with a center at the point (0,0) is the Einstein circle. The source radius is $r = 0.1$, the distance of the source center from the optical line is $d = 0.3$.

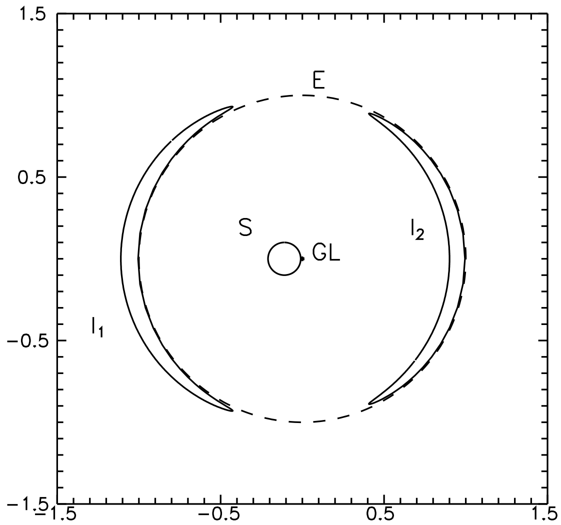

Figure 3: The source and the images which are distorted by point mass lens for distance of the source center from the optical line is $d = 0.11$.

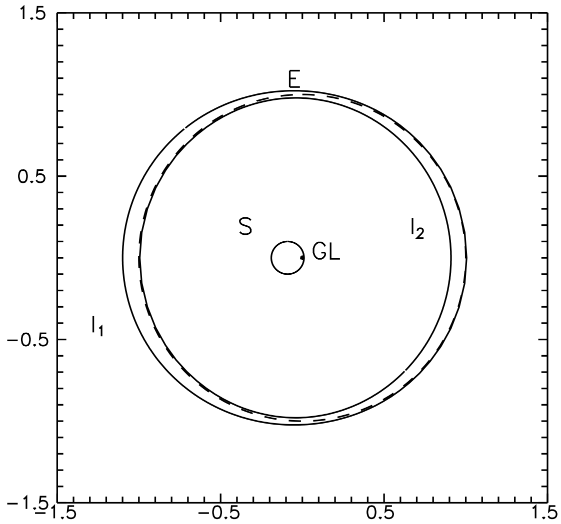

Figure 4: The source and the images which are distorted by point mass lens for distance of the source center from the optical line is $d = 0.09$. There are the same notations as in captions for Fig. 2 and Fig. 3. For the case the gravitational lens projection is inside the source projection on the celestial sphere.

$$\frac{dx^{\pm}}{dy} = \frac{1}{2} \pm \frac{y}{\sqrt{4+y^2}} = \frac{1}{2} \pm \frac{y}{2} + O(y^2), \tag{12}$$

so it is clear that size of the images less than the source size in two times in radial direction.

If we consider the case, when the source lies on the optical line (the gravitational lens and the observer are on the line also) the the image is the so-called Einstein circle, which was discussed by Chwolson [20] and by Einstein [23]. It is necessary to note that Einstein wrote: *"Of course, there is no hope of observing this phenomenon directly".* But it would be more correct to say in case that it is impossible to observe the circle, since the circle appearance is the structural stability of the lens mapping, according to Arnold's terminology [32]. Really, if we consider infinitesimal changes of parameters of the gravitational lens mapping then the circle disappears and there are only two separate images of the point source [33,34,35]. If we consider extended source then the appearance of the ring is possible in principle, but it is less probable event than appearance of arcs.

Let us define the magnification factor of gravitational lens. It is necessary to remark that since magnification factor is greater than unity, it is possible to use gravitational lensing for the name of the light deflection phenomenon and the gravitational lens for the name of the gravitating body. If we change linear variables by angular ones, namely, we consider $\boldsymbol{\theta} = \boldsymbol{\xi}/D_0, \quad \boldsymbol{\beta} = \boldsymbol{\eta}/D_s$, then the solid angle of the source is less than the sum of solid angles of images and the factor characterises the magnification factor of the gravitational lens. Thus if the solid angle $\Delta\omega_0$ corresponds to the source and the solid angle $\Delta\omega$ corresponds to images, then the magnification factor is equal

$$\mu = \frac{\Delta\omega}{\Delta\omega_0} = |\det d\boldsymbol{\beta}/d\boldsymbol{\theta}|^{-1} = |\det d\boldsymbol{y}/d\boldsymbol{x}|^{-1} \tag{13}$$

since

$$\boldsymbol{y} = \boldsymbol{\beta}/\theta_0, \quad \boldsymbol{x} = \theta/\theta_0.$$

Let us choose the frame of reference so that $x_2 = y_2 = 0$. We calculate the Jacobian matrix $\boldsymbol{x}_\pm \mapsto \boldsymbol{y}$ (at the point $\boldsymbol{y}= (y_1, 0)$), which is called
$\mu_\pm = |\det d\boldsymbol{x}^\pm/d\,\boldsymbol{y}|$ In the case only diagonal elements of the Jacobian matrix are not vanished, namely

$$\frac{\partial {x_1}^\pm}{\partial y_1}\,|_{(y_1,0)} = \frac{1}{2}\left(1 \pm \frac{y_1}{\sqrt{4+{y_1}^2}}\right), \tag{14}$$

$$\frac{\partial {x_2}^\pm}{\partial y_2}\,|_{(y_1,0)} = \frac{1}{2}\left(1 \pm \frac{\sqrt{4+{y_1}^2}}{y_1}\right), \tag{15}$$

so, we have

$$\mu_\pm = \frac{1}{4}\left(\frac{y_1}{\sqrt{4+{y_1}^2}} + \frac{\sqrt{4+{y_1}^2}}{y_1} \pm 2\right), \tag{16}$$

$$\tag{17}$$

and it is clear that the ratio of the magnification factors of different images is defined by the following expression

$$\frac{\mu_+}{\mu_-} = \left(\frac{\sqrt{4+{y_1}^2}+y_1}{\sqrt{4+{y_1}^2}-y_1}\right)^2. \tag{18}$$

$$\tag{19}$$

Let us consider the limiting magnification factors $\mu_\pm$ and their ratios for $y_1 \to 0$ (similarly to [22], [36])

$$\mu_+ = \frac{1}{2y_1} + \frac{1}{2} + O(y_1), \tag{20}$$

$$\mu_- = \frac{1}{2y_1} - \frac{1}{2} + O(y_1), \tag{21}$$

$$\frac{\mu_+}{\mu_-} = 1 + y_1 + O({y_1}^2). \tag{22}$$

We obtain also for $y_1 \to \infty$

$$\mu_+ = 1 + {y_1}^{-4} + O({y_1}^{-6}), \tag{23}$$

$$\mu_- = {y_1}^{-4} + O({y_1}^{-6}), \tag{24}$$

$$\frac{\mu_+}{\mu_-} = {y_1}^{4} + O({y_1}^{0}). \tag{25}$$

However, it is necessary to remark the point mass lens model is very instructive, but it is possible to use it with care [22,36], since in astrophysical conditions either the angular distance between images is very small for a direct observation (the Einstein angle is very small as for samples of microlensing) or gravitational lens has a great mass (and a great size also), so it is impossible to accept the point mass lens model (as for the first gravitational lenses [28]).

Bibliographical remarks

There is a very detailed and an interesting description of the historical aspects of the gravitational lenses in the book of Schneider and others [22], in the book of Bliokh and Minakov [5] and in Wamsganss's thesis [37]. Soldner's biography and English translation of his classical article are in the paper [8]. Some historical remarks, Soldner's derivation of expression (1), brief derivation of Einstein expression and basic concepts of gravitational lenses are in the introductory paper [36]. The list of gravitational lenses and known giant arcs are presented in the review of Refsdal and Surdej [28]. The modern analysis of observational microlensing data and their theoretical interpretation are given in the recent paper [38].

Recently, there were appeared very comprehensive review by Narayan and Bartelmann [39], and introductory lectures on gravitational lensing by Schneider [40]. The papers are very interesting for everybody who studies the gravitational lens subject.

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