RELATIVISTIC MECHANICS WITH RETARDED INTERACTIONS

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1. Introduction

There it little doubt now that interactions between any separated physical objects are always transmitted (mediated) by some continuous mediators: by elastic medea in the nonrelativistic case or by fields and field-like objects (e.g. metric tensor in GR) in the relativistic case. In both cases, the interaction transfer between distant objects takes some time and the energy-momenta lost by one object reaches the other object with some retardation: with acoustic delay caused by the limited speed of sound in the elastic medium, or with relativistic retardation caused by the limited speed of light. While the energy-momenta are en-route, they are stored by the mediator of interactions.

The standard approach to formulation of mechanical problems with distant interacting objects usually suggests the dilemma: either to consider the transmitter of interactions in its full details with explicit description of the excitation and propagation of corresponding waves, or to neglect the retardation effects and the inertial properties of the transmitter completely. In the first extreme, one may be helped by the acoustics and the field theory, in the second extreme, by the apparatus of the Hamiltonian or Lagrangian formalisms.

However, there is a large intermediate area of problems, where the retardation of interactions and the inertia of the mediator cannot be neglected, and the formulation and solution of the full-detail wave-propagation task is too difficult or impossible. It is necessary to develop the methods of the formulation, analysis, and solution of the mechanical problems, where the retardation is essential and the explicit consideration of waves should be avoided.

In case of the relativistic processes with strongly interacting particles, the need for such methods is especially acute, since the very existence of the classical analogues of the corresponding quantum fields (e.g. of the quantum quark-gluonic field) is still problematic.

As the answer to this and similar needs, the construction of the various relativistic many-times (MT) models [1-5] and direct interaction (RDI) theories, classical and quantum [6-24], and of predictive mechanics [27-30], was undertaken.

The considered relativistic MT models start from general explicitly Poincare-invariant Newton-type equations of motion, but then, for various reasons, sacrifice the relativistic causality in favour of possibility of the Lagrangian and Hamiltonian formulations with corresponding conservation laws and in favour of getting the closed-orbit solutions. They generally lead to integral conservation laws, in particular, to conservation laws [5] involving integration over time to $+\infty$, which are essentially post-mortem relations. The inertial properties of the mediating field ("inertia of interactions") is not considered explicitly, though may be taken into account indirectly by its inclusion into the expression for the canonical momenta.

RDI theories take into account the inertia of the mediator and the relativistic relation between energy and mass. They are able to describe elastic scattering and the orbital motion. The common idea of all RDI theories is to replace the actual retarded interactions by some "equivalent" instant interaction compatible with the structure of the Poincare group and able to give the same scattering and stationary states predictions as the field theory. In most of these theories, the interaction is fixed by one scalar function interpreted as the full (rest) mass of the particle system. In some versions (based on the Pfaff equations [15]), a second scalar function is present in interaction terms.

The structure of the Poincare group permits different ways to separate the internal motion, to introduce interaction terms, and to select the foliation of the space-time. These ways, called the forms of the relativistic dynamics [6,7,8,12,14,19,24], correspond to different physical assumptions about the inertial properties of the mediator of interactions. For example, the point form of dynamics corresponds to the assumption that the mediator has the 4-momentum, but has no spin (i.e. no momenta corresponding to the spatial and the Lorentz rotations). The instant form of dynamics means that the mediator has the energy and the momenta corresponding to the Lorentz rotations, and has no other momenta. These and other forms of dynamics were found to be equivalent [14] in the sense that, for the same interactions fixed by expressions for the total mass, they predict the same scattering crossections and the bound state energies in the quantum case. (By means of the Lorentz-invariant classical limit [18], the similar statements about the canonical scattering transformations and mass/period ratios for finite motion solutions can be obtained for the classical case.)

The main difficulty of the RDI theories is that they give the trajectories of particles in some phase spaces, but not in the physical Minkowski space M4 with unambiguously measurable coordinates. It was first noticed in the Hamiltonian formulation, where the famous "no-interaction" theorem [8,26] excluded the coincidence between the canonical coordinates q and the Minkowski space coordinates x. The absence of coincidence itself is not a major problem since some functions X of the canonical variables Q=(p,q) and of the evolution parameter T may be found that are transformed as points x of the Minkowski space, and one may try to interpret these functions as space-time coordinates of particles [20-25]. The real physical problem is that in case of several interacting particles the interaction terms cannot be expressed through the differences of coordinates X, so the strength of forces between particles cannot correspond to particle closeness in M4. In these conditions, it is difficult to understand how coordinates X can be measured, and, if measured, why the results of measurements by means of different probing particles will be the same. This makes the choice between different possible definitions of X uncertain and their interpretation ambiguous. The definition of interaction with an external field given in M4 becomes a problem as well.

These difficulties may look differently in different formulations of RDI theories, but everywhere the definition of measurable coordinates X and of interaction with external fields are a problem. Since the origin of these problems is the violation of relativistic causality [31], some reflection of these problems should be expected in other approaches disregarding the causality. Especially serious these problems become in case of many particles, when the system closed (in the terminology of Havas) contains subsystems which are open with respect to interactions with the rest of particles.

These difficulties limit the domain of reliable practical applicability of the RDI theories and other approaches, disregarding the relativistic causality, to the estimation of time-independent values like S-matrix elements and the energies of the bound states (and of their classical analogues) and make these approaches inadequate when external fields are present.

In the present paper, we consider a causal version of relativistic MT mechanics (with retarded interactions only) and develop it making a step toward a field theory by introducing into the equations of particle motion a new mediator of interactions, having, besides inertia, some field properties. The introduced mediator of interactions is able, like fields, to pass interactions with causal retardation, but is, like the potential, unable to excite its internal degrees of freedom. Such mediator can be considered as a very simplified model of a field, so we will call it a reduced field (RF). While acting on particles, RF is accumulating from particles and passing to particles, besides the energy, all kinds of linear and angular momenta, so its state may be described by the set of 10 values corresponding to 10 generators of the Poincare group.

With respect to its inertial properties, RF is similar to the most general mediators of the RDI theories. However, the forces, produced by RF, unlike forces of RDI theories, depend on retarded positions, velocities, and, possibly, accelerations of particles, as the Lienard-Wiechert forces do in the electromagnetic theory. Besides the dependence on the particle coordinates and velocities, the RF forces, generally, depend on 10 values describing the state of RF. The last dependence is essential for the existence of models describing elastic scattering and periodic (precessing) orbital motion of strongly interacting particles.

The relativistic classical mechanics with retarded interactions transmitted by RF is a many-time theory of the Newtonian type. However, the introduction of the description of the state of RF and the completion of the equations of motion by the equations for RF evolution, eliminates the difficulties with the energy-momenta conservation laws and makes the theory logically closed. The respecting of the causality principle removes any problems with the measurability and interpretation of the X coordinates. In this respect, the mechanics with RFs is as fundamental as the nonrelativistic Hamiltonian mechanics, but is more general and more accurate at high velocities. It is formally independent from the classical and quantum field theory, though systematically uses the field picture as the source of inspiration.

The properties of RF and of mediators of other classical theories can be illustrated by table:

Mediators:	Potential	Rel. potential	Reduced field	:::	Field
energy	Yes	Yes	Yes	Yes	Yes
momenta	No	Yes	Yes	Yes	Yes
retardation	No	No	Yes	Yes	Yes
excitations	No	No	No	:::	Yes

The dots here stand for the mediators intermediate between RF and a field and describing explicitly some of the field excitations.

The main advantage of the explicit use of retarded arguments in the interaction terms is the possibility to formulate equations of motion in terms of the physical (measurable) Minkowski space coordinates and combine freely the interparticle interactions with the interactions of particles with external fields given in the Minkowski space. The main disadvantage of such theory is that it falls outside the well-developed Hamiltonian and Lagrangian formalisms in their usual forms (its possible relations with various generalizations [32-35] of these formalisms remain unclear). Due to this circumstance the standard methods of quantization based on the Hamiltonian and the Lagrangian formulations do not work for RF and, at present, we may consider the classical version of the RF theory only.

The equations of motion in case of retarded interactions belong to the domain of the theory of differential equations with deviated arguments [36-38]. The solutions of such equations show, generally, more complicated behaviour than the solutions of the equations without retardation. They often demonstrate various instabilities similar to instabilities known in the theory of automatic regulation with retarded feedback. One of the causes of possible instabilities of solutions is the absence of an explicit lower bound of the energy of RF. To obtain physically interesting models with stable particle motion, one has to choose carefully the magnitude and direction of forces and their dependence on the state of RF. The first concern of the RF theory is the description of the family of forces leading to stable solutions.

In [39], the linear motion of two particles interacting via RF was analyzed and it was shown that certain choice of the dependence of forces on the state of RF makes the scattering elastic at arbitrary high energies.

In the present paper, we, besides the general introduction into the mechanics with RFs, analyze the planar motion of two particles interacting with retardation and construct retarded forces leading to an elastic scattering of the particles, and to stable orbital motion in case of attraction. The main attention in the paper will be paid to the correspondence between the structure of forces and the physical intuitive picture of particles interacting through fields coating the particles.

NOTATIONS

$x = (t; \mathbf{x})$; $p = (p_0; \mathbf{p})$ denote the coordinate and momentum of the particle, $m = |p|$ is its mass, $h = p{=}m$ is its 4-velocity. denotes the proper time: $d \; = dt{=}h_0$. Derivatives with respect to are denoted by dot. The scalar products $x \cdot p$ will imply the metric g=diag(1,-1,-1,-1). Vector $R_i = -R_j = x_i - x_j$, $H = h_i + h_j$.

The index ^{ret} means the value at the retarded position $x_j^{\mathrm{ret}}(x_i)$ of particle j with respect to the position of particle i. $R_i^r = x_i - x_j^{\mathrm{ret}}$ is the null vector: $(R^r)^2 = 0$. Index ^{r} means that one of the arguments of a two-argument quantity is retarded.

Scalar $T_i = R_i^r \cdot h_i$ is called the retardation time, scalar $D_i = R_i^r \cdot h_j^{\mathrm{ret}}$ plays role of a distance between particles $i; j$ in the rest frame of particle j.

$s = x \wedge p$ means the antisymmetric tensor (spin) with elements $s_{ab} = x_a p_b - x_b p_a$. The scalar product of vector y and tensor s means $y \cdot s = -s \cdot y = y \cdot x\, p - y \cdot p\, x$.

2. Equations of motion

The state of the system of particles and of RFs is fixed by the coordinates and momenta of particles $x; p$ and by the states U of RFs.

We consider the simplest possible case, when particles are unchangeable and have no self-interactions. Then the masses of particles are constant and the state U of each reduced field may be fixed by its 4-momentum Q and spin S describing the internal rotational state of RF. Antisymmetric tensor S has 6 independent elements, so the state U has 10 independent components. RF is not a particle and has no x-coordinates.

Like the nonrelativistic potential which may describe paired, triple, or multiparticle interactions, vanishing when any of the particles in the subset is far away and depending, correspondingly, of two, three, or more particle coordinates, RF may as well describe paired, triple, and so on, interactions and depend on two or more points x_i, or, since points x_i are functions of times $_i$, depend of two or more times $_i$.

We will define $Q; S$ through functionals $q; s$ describing the momentum and angular momentum contributions to RF from each particle and depending on one time only. (In general case, when the self-interactions are admitted, the values $q; s$ may get the physical status of states of RFs responsible for the self-interactions. Here we use them as auxiliary quantities.) We put

$$\mathrm{Q}_{ij\ldots}(\ _i;\ \ _j;:::) = \mathrm{q}_{i,\ldots}(\ _i) + \mathrm{q}_{j,\ldots}(\ _j) + :::;$$

$$\mathrm{S}_{ij\ldots}(\ _i;\ \ _j;:::) = \mathrm{s}_{ij\ldots}(\ _i;\ \ _j;:::) - \mathrm{X}_{ij\ldots} \wedge \mathrm{Q}_{ij\ldots};$$

where

$$\mathrm{s}_{ij\ldots}(\ _i;\ \ _j;:::) = \mathrm{s}_{i,\ldots}(\ _i) + \mathrm{s}_{j,\ldots}(\ _j) + :::$$

and $\mathrm{X}_{ij\ldots} = \mathrm{X}(\ _i;\ \ _j;:::)$ is some collective ("center of mass") coordinate of the subset of particles $\mathrm{ij}:::$. The subtraction of term $\mathrm{X} \wedge \mathrm{Q}$ makes tensor S translationally invariant.

In case of paired interactions, RF states U depend on two times and values $\mathrm{Q};\mathrm{S}$ are

$$\mathrm{Q}_{ij} = \mathrm{q}_{i,j} + \mathrm{q}_{j,i};$$

$$\mathrm{S}_{ij} = \mathrm{s}_{i,j} + \mathrm{s}_{j,i} - \mathrm{X}_{ij} \wedge \mathrm{Q}_{ij}:$$

In this case, the equations of motion of particles $\mathrm{i} = (1;:::;\mathrm{N})$ and RFs are

$$\mathrm{dx}_i = \mathrm{d}\ _i\mathrm{p}_i{=}\mathrm{m}_i; \tag{1}$$

$$\mathrm{dp}_i = \mathrm{d}\ _i(\mathrm{F}_i + \sum_j \mathrm{F}_{ij}); \tag{2}$$

$$\mathrm{dq}_{i,j} = -\mathrm{d}\ _i\mathrm{F}_{ij}; \tag{3}$$

$$\mathrm{ds}_{i,j} = -\mathrm{d}\ _i\mathrm{x}_i \wedge \mathrm{F}_{ij}; \tag{4}$$

where one-index force F_i describes interaction with an external field and two-indeces forces F_{ij} describe interactions through reduced fields. Force F_i depends on $\mathrm{x}_i;\mathrm{p}_i$ only and does not contribute directly to the momentum and spin of RFs.

In case of multiparticle forces, the equations of motion are similar, only the simple index j is replaced by appropriate collective index $\mathrm{J} = (\mathrm{j};:::)$.

Unlike the Hamiltonian theory, the forces F in this theory are not defined as the gradients of the mediator energy, but, conversely, the differentials of the state U are defined through the forces which should be specified as functions of $\mathrm{x};\mathrm{p};\mathrm{U}$ respecting the causality principle.

This principle, generally, allows the dependence of force F_{ij} on any functional of the present trajectory $\mathrm{x}_i(\mathrm{t}_{\mathrm{old}})$, $\mathrm{t}_{\mathrm{old}} \leq \mathrm{t}$, and of retarded parts of other trajectories $\mathrm{x}_j(\mathrm{t}_{\mathrm{old}})$: $\mathrm{t}_{\mathrm{old}} \leq \mathrm{t}^{\mathrm{ret}}$, $(\mathrm{x}_i - \mathrm{x}_j(\mathrm{t}_{\mathrm{old}}))^2 \leq 0$. In particular, the forces may depend on the last RF state $\mathrm{U}_{ij}(\ _i;\ \ _j^{\mathrm{ret}})$ or on an earlier RF state $\mathrm{U}_{ij}(\ _i^r;\ \ _j^{\mathrm{ret}})$, where $\ _i^r < \ _i$ and is defined through other variables in some Lorentz-invariant way. Typically, the forces have form

$$\mathrm{F}_{ij} = \mathrm{F}_{ij}(\mathrm{x}_i;\mathrm{p}_i;\mathrm{x}_j^{\mathrm{ret}};\mathrm{p}_j^{\mathrm{ret}};\dot{\mathrm{p}}_j^{\mathrm{ret}};\ddot{\mathrm{p}}_j^{\mathrm{ret}};\mathrm{U}_{ij}(\ _i;\ \ _j^{\mathrm{ret}});\mathrm{U}_{ij}(\ _i^r;\ \ _j^{\mathrm{ret}});:::):$$

If we had admitted the self-interactions of particles, the forces would depend on the functionals $\mathrm{q};\mathrm{s}$ separately. In the simplest case considered in this paper, the dependence of forces on RF states is restricted to the dependence on U only.

Equations of motion (1-3) do not depend on the assumption that particles are unchangeable. This assumption puts the resrtriction on forces: to keep the masses m constant, all forces should be orthogonal to p:

$$\mathrm{p} \cdot \mathrm{F} = 0: \tag{5}$$

The initial data for the equations of motion should include the pieces of trajectories of particles and of evolution of states U (or functionals q; s) during some period in the past. This period should be long enough to contain all the retarded moments $_j^{\mathrm{ret}}(\mathrm{x}_i)$ for all the present points x_i. The initial trajectories and RF states need not satisfy any equations and may be arbitrary. The freedom of their choice physically corresponds to the freedom of choice of forces from external fields in the past.

The equations of motion are stating that, if some energy-momentum or angular momentum is given to particles by RF forces, the same amount of energy-momenta is taken from RFs. So, if we define the total energy-momenta and the total angular momentum of the isolated system of particles and of RF as

$$\mathrm{P}_{\mathrm{total}} = \sum_i \mathrm{p}_i + \sum_{ij} \mathrm{Q}_{ij};$$

$$\mathrm{s}_{\mathrm{total}} = \sum_i \mathrm{x}_i \wedge \mathrm{p}_i + \sum_{ij} \mathrm{s}_{i,j}$$

their conservation is a trivial consequence of the equations of motion. Indeed, in the absence of an external field the differential of the total 4-momentum obviously vanishes

$$\mathrm{dP}_{\mathrm{total}} = \sum_i \mathrm{dp}_i + \sum_{ij} \mathrm{dq}_{i,j} = 0:$$

Since (due to identity $\mathrm{dx} \wedge \mathrm{p} = 0$) the differential of the angular momentum of the particle is

$$\mathrm{d}(\mathrm{x}_i \wedge \mathrm{p}_i) = \mathrm{x}_i \wedge \mathrm{dp}_i = \mathrm{x}_i \wedge \sum_j \mathrm{F}_{ij} \mathrm{d}\ _j;$$

the differential of the total angular momentum vanishes as well

$$\mathrm{ds}_{\mathrm{total}} = \mathrm{d}(\sum_i \mathrm{x}_i \wedge \mathrm{p}_i) + \sum_{ij} \mathrm{ds}_{i,j} = 0:$$

In particular, if we consider the collision of particles and the reduced field is the same before and after the collision ($\mathrm{Q}_{\mathrm{initial}} = \mathrm{Q}_{\mathrm{final}}; \mathrm{s}_{\mathrm{initial}} = \mathrm{s}_{\mathrm{final}}$), the energy-momenta of the particle system is conserved in the collision. (If $\mathrm{Q}_{\mathrm{initial}} = 0$, angular momentum $\mathrm{s}_{\mathrm{initial}}$ of RF may be replaced by its spin $\mathrm{S}_{\mathrm{initial}}$.)

In exceptional cases, when both the energy and the squared mass of the difference $\Delta\mathrm{U} = \mathrm{U}_{\mathrm{final}} - \mathrm{U}_{\mathrm{initial}}$ are positive, one may try to interpret $\Delta\mathrm{U}$ as the energy-momenta of the radiated field. It can be done, for example, for the electromagnetic interaction described by the Lienard-Wiechert forces with the radiation friction term (compare the method of calculating the radiation in [42]). But usually, the nonzero value of

ΔU has no clear interpretation and indicates that the reduced field description of the mediator of interaction is inadequate and that more detailed description of the mediating field, including its excitation and radiation properties, is needed.

The condition $\Delta U = 0$ of elastic scattering limits the domain of complete logical consistency of the mechanics with RFs. For this reason, the existence of the family of forces, for which ΔU is exactly zero in a wide region of relative velocities of particles, is a crucial question of the theory. In this paper, we explicitly construct a large family of such forces, exploiting the possible dependence of forces on the state U.

The description of RF by quantities U does not tell how the field is distributed in space, it only tells how much energy-momenta the field contains. In this respect, it is similar to the nonrelativistic potential which tells how much energy the field has, but not where in the space it is. However, the dependence of forces on the state U indirectly indicates, how close to the particles RF is accumulated.

The equations of motion are time-asymmetrical, and so are the solutions of these equations. The equations of motion are asynchronous. They can be integrated for each particle i together with relevant functionals $q_{i,\ldots}; s_{i,\ldots}$ independently of other particles and of other functionals while the retarded moments $^{\mathrm{ret}}(x_i)$ are contained in the known parts of trajectories. In these respects, the solutions of retarded equations of motion are qualitatively different from the solutions in the Hamiltonian theory. However, certain choice of dependence of forces on U may make solutions rather close to those of the Hamiltonian theory at low velocities.

Though the equations of motion are asynchronous, one may as well consider proper times as functions of some common evolution parameter t and solve equations synchronously, if it is convenient. The solution is identically the same in both cases.

The choice of evolution parameter is, generally, subject only to the condition that all vectors $x_i({}_i(t)) - x_j({}_j(t))$ are always space-like. Let us consider "synchronized" equations of motion.

Denoting the evolution parameter by t and the time derivative d=dt by prime, we may write equations of motion as

$$x_i^0 = {}_i^0 h_i;$$

$$p_i^0 = {}_i^0(F_i + \sum_j F_{ij});$$

$$q_{i,j}^0 = - {}_i^0 F_{ij};$$

$$s_{i,j}^0 = - {}_i^0 x_i \wedge F_{ij}:$$

From these equations one can extract equations for the full state of RF. Summing the equations of motion for q, we obtain

$$Q^0 = -F^{+}; \tag{6}$$

where

$$F^{+} = F_i \;{}_i^{0} + F_j \;{}_j^{0}$$

is a sum of forces at time t.

The definition of RF spin depends on the choice of collective coordinate $X_{\dots}$. Let $X_{ij} = (x_i + x_j)$=2. Then, the time derivative of RF spin for two particles takes form

$$S_{ij}^{0} = -[R_i \wedge F^{-} + (h_i \;{}_i^{0} + h_j \;{}_j^{0}) \wedge Q]{=}2; \tag{7}$$

where

$$F^{-} = (F_{ij} \;{}_i^{0} - F_{ji} \;{}_j^{0}):$$

Generally, equations (6-7) are coupled with equations for differences $q_{i,j} - q_{j,i}$ entering force F^{-}, if forces depend on U. This coupling may be eliminated, if the correspondence between two arguments of $U_{ij}(\;{}_i^{r};\;{}_j^{\mathrm{ret}})$ is reciprocal:

$$x_i^{r} = \;(x_j^{\mathrm{ret}}); \quad x_j^{\mathrm{ret}} = \;(x_i^{r}):$$

Then both forces $F_{ij}; F_{ji}$ will depend on the states on the same family of hyperplanes and the differences $q_{i,j} - q_{j,i}$ will drop out from force F^{-}.

The correspondence (synchronization) between two arguments of U must be unambiguous, monotonous in time, and Lorentz-invariant. It should, for good convergence to the nonrelativistic limit, place the particles as symmetrical as possible in each other field, that is to make the distances $D_i; D_j$ close to each other and the retardation times $T_i; T_j$ close to each other. In [39], the exact equality of retardation times $T_i; T_j$ was used as synchronization condition of after-scattering trajectories. In case of large accelerations, such condition is ambiguous. We will use another condition

$$R_i \cdot R_i^{r} = R_j \cdot R_j^{r} \tag{8}$$

satisfying all the above requirements in the general case. Differentiating (8), we get the relation between $\;{}_i^{0};\;{}_j^{0}$:

$$\;{}_i^{0}K_i = \;{}_j^{0}K_j; \tag{9}$$

where

$$K_i = h_i \cdot (R_i^{r} + R_j^{r}) + R_i \cdot (h_i - h_j^{\mathrm{ret}} d\;{}_j^{\mathrm{ret}}{=}d\;{}_i):$$

The derivative $d\;{}_j^{\mathrm{ret}}{=}d\;{}_i$ can be found by differentiating $(R_i^{r})^2 = 0$. It gives

$$d\;{}_j^{\mathrm{ret}}{=}d\;{}_i = T_i{=}D_i:$$

In the choice of evolution parameter t, it is advantageous (though not obligatory) to define it to have the same value for points $x_i^{r}; x_j^{\mathrm{ret}}$. In case of synchronization (8), evolution parameter t, according (9), may be defined by

$$dt = kd\;{}_iK_i;$$

where k is a scale factor. Since at small accelerations K_i is close to $T_i + T_j$, it is convenient to put $k = 1{=}(T_i + T_j)$, what finally gives

$$t = \int \frac{K_i d \ _i}{(T_i + T_j)}: \tag{10}$$

Then, when accelerations are small, ${}^0_i \approx {}^0_j \approx 1; \qquad (x_i - x_j) \cdot H \approx 0$: In such cases, more simple synchronization

$$(x_i - x_j) \cdot H = 0 \tag{11}$$

may be used.

3. Forces and the Field Picture

The freedom of the choice of forces F from RFs is larger than the freedom of the choice of the interaction potential in the nonrelativistic Hamiltonian or Lagrangian mechanics and in RDI theories. But the subset of "good" forces leading to physically reasonable solutions of equations of motion is relatively narrow and its selection is less easy than selection of "good" Hamiltonians. To find the simplest families of "good" forces, we will appeal to the naive physical field picture of interacting particles.

We may imagine particles as some fields with singularities moving with accelerations, if the stable (symmetrical) configuration of the field near a singularity is perturbed by proximity of other singularities. In some approximation, the field with several singularities may be represented as the sum of unperturbed fields of separate singularities and some (nonlinear) corrections. The separate singularities with their unperturbed fields are described in equations of motion as particles in states $x; p$. The energy-momenta of the interference terms and of the field corrections are described by the state U of RF. In case of two particles, RF energy-momenta is just the difference between the initial energy-momenta of the field of two infinitely far separated singularities, and the energy-momenta of the field with two singularities at finite distance. In case of more particles, the partial contributions of different fields and forces are singled out as separate RFs.

Clearly, the separation of the field into "particles" with constant masses and some perturbation used as a "mediator of interaction", implied in all classical theories, is conventional and is not always justified. In particular, it is not quite adequate when the energy of RF is negative and comparable with the masses of (free) particles. We keep to the assumption $m =$const for simplicity of equations.

When particles are far separated, the perturbation of their fields are small, and the force acting on particle i from the side of particle j is proportional to the unperturbed field of particle j at point x_i. This force depends on the positions and velocities of particles, but does not depend on the state U of RF, or on accelerations. It usually dominates, especially at low velocities. We shall call it a primary force and denote F^P. The Coulomb force is an example of a nonrelativistic limit of a primary force.

When particles are close and move fast, the perturbation of their fields creates additional forces. These forces are more complicated than F^P and may depend (besides positions and velocities) on accelerations of particles and on the state of RF. There are three simplest possibilities.

1) The field perturbations may fly away as a radiation and carry away their energy-momenta. Then they may produce forces only at the moment when the radiation wave passes the particle. The amplitude of the wave from particle j and the corresponding force acting on particle i should depend on the acceleration of particle j, and, hence, on the force acting on particle j. Other words, the force acting on one particle creates, after a proper retardation, the force acting on other particle. We shall call such forces echo forces and denote them F^E. The part of the Lienard-Wiechert force proportional to acceleration is an example of an echo force. (Interactions, dependent on accelerations, were considered in the Lagrangian formulation as well [35]).

2) The field perturbations may accumulate somewhere near the particles. Since they may not accumulate indefinitely, and since the perturbed field configurations are generally unstable, they should return their energy-momenta back to the particles by means of some forces. This process can be interpreted as a decay of field distortions. We shall call the corresponding forces decay forces and denote them F^D. Decay forces should depend on the state U of RF.

3) The field perturbations may accumulate somewhere near the particles, but before their dissipation in space near the particles the perturbation wave may reach the second particle and pass essential part of its momenta to that particle. It may happen, if the field of two particles concentrates near a line connecting particles and behaves as an elastic string. The wave along a string may create relatively strong echo force F^E, dependent both on the accelerations of particles and on the state of RF.

In the next three sections, we shall construct the simplest forces $F^P; F^D; F^E$ having the properties suggested by the field picture. Combining these forces, one may build models of relativistic physical systems of interacting particles with desired behaviour, in particular, systems with elastic scattering and with finite motion. The exact account of electromagnetic interactions (by means of the Lienard-Wiechert forces) can by made as well.

4. Primary forces

Let us suppose that the field belonging to a particle and moving with it becomes spherically symmetric for the isolated particle and rotationally symmetric around the line connecting the pair of interacting particles, if the particles are at rest long enough. Then, the force in the static case is central and can be written as

$\mathbf{f} = \mathbf{R}v'(|\mathbf{R}|)/|\mathbf{R}|$, where v' means the derivative of function v with respect to its argument. Arbitrary scalar function v is the static potential related with the field.

In the theory with causal retardation, the field may not change immediately everywhere, when particles accelerate. Let particle i be at point x_i and particle j be moving. If particle j accelerates, its field at point x_i continues to move during the retardation time T in the direction of velocity $h_j^{\rm ret}$, which the particle j had in its retarded position $x_j^{\rm ret}(x_i)$. Hence the force, acting on particle i at point x_i is directed not toward point $x_j^{\rm ret}$, but toward the extrapolated position

$$x_j^{\rm ext} = x_j^{\rm ret} + h_j^{\rm ret}\frac{T}{C_{ij}^r};$$

where particle j would arrive at time t if it were moving without acceleration. In the last expression, $C_{ij}^r = h_i \cdot h_j^{\rm ret}$ is the ratio of the differentials of the time in the rest frame of particle i (where T is defined) and of the time in the rest frame of particle j. So, the force acting on particle i has not the direction $R = x_i - x_j^{\rm ret}$, but the direction

$$R_i^{\rm ext} = x_i - x_j^{\rm ext}:$$

It is easy to check, that $h_i \cdot R^{\rm ext} = 0$ as it is required by the condition of constant particle mass: $p \cdot F = 0$.

As an illustration of this, let us take the Lienard-Wiechert force $F_{\rm EM} = e_i h \cdot \mathcal{F}$, where e_i is the charge and $\mathcal{F}$ is the tensor of the electromagnetic field

$$\mathcal{F} = e_j R^r \wedge (h \wedge R^r[R^r \cdot \dot{h}^{\rm ret} - 1]/D - \dot{h}^{\rm ret})/D^2;$$

where $D = (x_i - x_j^{ret}) \cdot h_j^{ret}$.

The primary force (containing no accelerations), is

$$F_{\rm EM}^P = -e_i e_j h \cdot (R^r \wedge h^{\rm ret})/D^3 = R^{\rm ext} e_i e_j h \cdot h^{\rm ret}/D^3$$

and is directed just along the vector $R^{\rm ext}$. The "echo" part of the Lienard-Wiechert force which is proportional to acceleration and produced by the wave of the synchrotron radiation, has different direction dependent on the distance.

Consider now the relation of the force F^P with the relativistic generalization of the static potential. The relativistic invariance leaves a large freedom in the choice of such generalization. However, the above assumption that each particle drags its own field means that the strength of the force acting on particle i from the field of particle j should depend on distance $D_i = (x_i - x_j^{\rm ret}) \cdot h_j^{\rm ret}$ in the rest frame of the field.

Making the simplest assumption that the energy increment $d\ {}_iF_i^P \cdot h_j^{\rm ret}$ of particle i in the rest frame of particle j depends only on the increment of distance D (and does not depend otherwise on velocity h_i of particle i), we come to relation

$$F_i^P \cdot h_j^{\rm ret} = -\dot{\bar{v}}_i; \qquad (12)$$

where the bar means that, according to the definition of the primary force, the terms with accelerations $\dot{h}$; $\dot{h}^{ret}$ are omitted:

$$\bar{\dot{v}}_i = \bar{\dot{D}}_i v' = (C^r_{ij} - T_i{=}D_i)v': \qquad (13)$$

Since $F^P = R^{ext}$, where is a scalar function, from (12) and relation $R^{ext} \cdot h^{ret} = D - T{=}C^r$, we obtain

$$F^P_i = -R^{ext}_i \frac{C^r_{ij}}{D} v': \qquad (14)$$

Returning to the primary electromagnetic force F^P_{EM}, one may see that

$$v'_{EM} = -e_i e_j{=}D^2; \qquad v_{EM} = e_i e_j{=}D;$$

as it could be expected.

Primary forces of the form (14) make it possible to introduce 4-vector $\hat{Q}$ related with potential v and describing the adiabatic approximation of the RF energy-momentum Q. Unlike vector Q, dependent on the history of motion, vector $\hat{Q}$ is defined as a function of particles variables only. In this respect, it resembles relativistic interaction Hamiltonians in RDI theories. The main assumed property of $\hat{Q}$ is that

$$\hat{Q}^0 \underset{a.a.}{=} Q^0; \qquad (15)$$

where prime means the derivative with respect to some evolution parameter, "a.a." means the adiabatic approximation (omission of acceleration terms), and where only the primary forces are taken into account.

Condition (15) permits different specific definitions of $\hat{Q}$ depending on the assumptions about the particle fields. The simplest assumption that the fields of two particles are similar corresponds to

$$\hat{Q} = H v \; ;$$

where is a normalization factor depending only on $h_i; h_j$, and the correspondence between the points of two trajectories is given by a symmetric condition (8).

Synchronization (8) leads, in a.a., to equalities

$$ {}^0_i = {}^0_j = 1; \quad D_i = D_j; \quad T_i = T_j; \quad C^r_{ij} = C^r_{ji}; \quad R^{ext}_i + R^{ext}_j = H \frac{2}{H^2}(D - T{=}C^r):$$

This and expression (14) for forces convert equation (6) into

$$Q^0 = H v' \frac{2}{H^2}(C^r - T{=}D):$$

Comparing Q^0 with $\hat{Q}^0 \underset{a.a.}{=} H v'(C^r - T{=}D)$, we finally obtain

$$\hat{Q} = H v \frac{2}{H^2}:$$

Evidently, the energy component $\hat{Q}_0 \to v$ in the low velocity limit. So, if the solutions of equations of motion will be stable and Q will remain close to $\hat{Q}$, we will, with primary forces (14), come to the usual potential interaction in the nonrelativistic limit.

Static potential may be used as well to define an approximation of spin

$$\hat{S} = \int x_1 \wedge \hat{q}_1 d\ _1 + \int x_2 \wedge \hat{q}_2 d\ _2 - \frac{x_1 + x_2}{2} \wedge \hat{Q};$$

where $\hat{q} = -\frac{R^{ext}}{r} v$ and $r = R^{ext} \cdot H = D - T{=}C^r$. Spin $\hat{S}$, generally, is not an adiabatic approximation of S, since vectors $\hat{q}$, generally, are not adiabatic approximations of q (though always $\hat{q}_i + \hat{q}_j = \hat{Q}$).

However, in the planar case, $\hat{S}^0 \underset{a.a.}{=} S^0$. Indeed, in this case in the absence of accelerations trajectories intersect and vector R^{ext} does not change its direction with time, so the normalized vector $\frac{R^{ext}}{r}$ remains constant. Hence, the time derivative of $\frac{R^{ext}}{r}$ is proportional to accelerations and vanishes in the adiabatic approximation. So, in this case,

$$\hat{q}^0 \underset{a.a.}{=} -\frac{R^{ext}}{r} v^0 = -\frac{{}_0R^{ext}}{D} C^r v' = q^0$$

and $\hat{S}$ becomes an adiabatic approximation of spin S.

5. Echo force

The echo forces, by definition, depend on the (retarded) accelerations of particles $A^{ret} = F^{ret}{=}m$. Such force may become important, if the mediating field concentrates near the straight line connecting the particles and ties particles together as a light elastic string. The simplest field mechanism of an echo force is just a recoil wave along a string which reaches with a retardation the other end and passes part of its energy-momentum to the other particle. The maximum force transferable in such a way is the part of A^{ret} orthogonal to p: $F^E_{max} = A^{ret} - h\,h \cdot A^{ret}$: Generally, the efficiency of the momentum transfer between the wave and a particle may be less than one, so

$$F^E = a^E(A^{ret} - h\,h \cdot A^{ret});$$

where coefficient $0 \le a^E \le 1$ may depend on the angles between $h; R^r$ and $R^r; h^{ret}$.

The simplest echo forces increase, generally, the difference between the action and reaction, increase the accumulation of the momenta of RF (especially, of the spin of RF), and make the motion of particles less stable. It is possible to construct formally more complicated echo forces, dependent on the sum of retarded values of accelerations of two particles and on the mentioned angles, which have the opposite effect. However, it is difficult to understand what processes in the mediating field could lead to such forces, so we will not consider them here.

6. Decay force

The retardation of interactions leads to accumulation of values U, especially of spin components. The accumulation of RF occurs since the sum of spatial parts of forces acting on two particles is not zero and the forces are not directed along a line connecting the particles (the newtonian principle "action equals reaction" is violated by retardation). However small, such accumulation would exclude solutions with periodic orbital motion even at low velocities.

The considerable accumulation of RF may lead to pathological results. In particular, the RF energy Q_0 may tend to $-\infty$ giving the positive infinite energy-momentum to particles, and may remain negative, when particles separate and cease to interact. Solutions with large RF may have strange attractors or become fully chaotic.

The accumulation of values U can be limited by decay forces F^D depending on the state U of RF. The paper [39] demonstrated that in case of motion along a line the appropriate decay forces linear in U can make RF to vanish with time and make the scattering elastic at arbitrary high collision energy. In this paper, we will consider similar decay forces for more complicated case of planar motion. We will introduce more general decay forces, that are making the state U to tend to some "equilibrium" value V, where $V = (V^q; V^s)$ is some (10 component) function of particle variables. One may interpret the difference $\mathcal{U} = U - V$ as some dynamically created distortion or excitation of the static fields and a slow process of diminishing of $\mathcal{U}$ as a decay of this excitation.

Function V may be arbitrary. For example, one may set the "equilibrium" value equal to the adiabatic one $V = \hat{U} = (\hat{Q}; \hat{S})$:

Let us consider equations of motion for RF distortion $\mathcal{U}$. Subtracting V^0 from both sides of equations (6)-(7) (and omitting obvious index $_{ij}$ of $\mathcal{S}$), one gets similar equations

$$\mathcal{Q}^0 = -(f_{ij}\ {}^0_i + f_{ji}\ {}^0_j); \tag{16}$$

$$\mathcal{S}^0 = -[R_i \wedge (f_{ij}\ {}^0_i - f_{ji}\ {}^0_j) + H \wedge \mathcal{Q}]=2 \tag{17}$$

where f is the remaining force.

In case of choice $V = \hat{U}$, the remaining force is $f = F - F^P = F + R_i^{\mathrm{ext}} C_{ij}^r v$'=D and does not contain most of the principle force. It can be written as $f = f^A + F^D$, where force f^A is proportional to accelerations and vanishes in adiabatic approximation and F^D is a decay force depending on the state of RF.

In case of choice $V = 0$, force $f = F$ and the full RF is considered as a distortion: $\mathcal{U} = U$.

We will consider analytically only the decay forces linear in distortion $\mathcal{U}$. Such decay forces may be symbolically written as $F^D = B\,\mathcal{U}^{(\mathrm{ret})}$, where B is a linear operator transforming the complex $\mathcal{U}$ of 4-vector and of tensor into 4-vector F^D and index $^{(\mathrm{ret})}$ means that some of the arguments of the state $\mathcal{U}$ are retarded.

Then equations (16-17) will be linear in $\mathcal{U}$ and may be symbolically written as

$$\mathcal{U}^0 = \mathrm{AB}\,\mathcal{U}^{(\mathrm{ret})} + \mathrm{A}\,\mathrm{f}^{A}; \qquad (18)$$

where A is a linear operator transforming 4-vector f^{A} into the complex $\mathcal{U}$ of 4-vector and of tensor.

Since operator (matrix) B maps the space of larger dimension to space of smaller dimension, operator (matrix) AB cannot be diagonal, and equations in system (18) are always coupled. To simplify their analysis we will use the freedom in the choice of force $\mathrm{F}^{D} = \mathrm{B}\,\mathcal{U}^{(\mathrm{ret})}$ so as to make equations as little coupled as possible.

First of all, we will use (moving) orthogonal tripod $\mathrm{H} = \mathrm{h}_i + \mathrm{h}_j; \quad \mathrm{h} = \mathrm{h}_i - \mathrm{h}_j; \quad \mathrm{y}$, where y is a space-like vector orthogonal to H; h, normalized by $\mathrm{y}^2 = -1$, and lying in the plane of motion. In addition to these vectors, it is convenient to define vectors

$$\bar{\mathrm{h}}_i = (\mathrm{C}\,\mathrm{h}_i - \mathrm{h}_j)=\mathrm{n}; \quad \bar{\mathrm{h}}_j = (\mathrm{C}\,\mathrm{h}_j - \mathrm{h}_i)=\mathrm{n}$$

where $\mathrm{C} = \mathrm{h}_i \cdot \mathrm{h}_j$ and $\mathrm{n} = \mathrm{C}^2 - 1$, which are orthogonal, respectively, to $\mathrm{h}_i; \mathrm{h}_j$ and normalized so that $\mathrm{h}_j \cdot \bar{\mathrm{h}}_i = \mathrm{h}_i \cdot \bar{\mathrm{h}}_j = 1$, and to define their combinations

$$\bar{\mathrm{H}} = \bar{\mathrm{h}}_i + \bar{\mathrm{h}}_j; \quad \bar{\mathrm{h}} = \bar{\mathrm{h}}_i - \bar{\mathrm{h}}_j$$

with properties

$$\bar{\mathrm{H}} \cdot \mathrm{H} = 1; \quad \bar{\mathrm{h}} \cdot \mathrm{h} = -1; \quad \bar{\mathrm{H}} \cdot \mathrm{h} = \bar{\mathrm{h}} \cdot \mathrm{H} = 0::$$

$$F_{ji}^{D} = a_0 \bar{h}_j^r \mathcal{Q}_{H^{rr}}^{rr} - b_0 y^r \mathcal{S}_{h^{rr} y^{rr}}^{rr} - a_h \bar{h}_j^r \mathcal{Q}_{h^{rr}}^{rr} - b_h \bar{h}_j^r \mathcal{S}_{H^{rr} h^{rr}}^{rr} + a_y y^r \mathcal{Q}_{y^{rr}}^{rr} + b_y y^r \mathcal{S}_{H^{rr} y^{rr}}^{rr}; \quad (20)$$

where coefficients $a_0; b_0; a_h; b_h; a_y; b_y$ determine the decay rates of different components of RF and are arbitrary within certain limits depending on retardation time.

The terms in (19),(20) have a simple interpretation.

The term with coefficient a_0 is a force returning the energy of RF back to particles.

The term with coefficient a_h is a force returning the momentum of RF along the relative velocity h back to particles.

The term with coefficient a_y is a force returning the momentum of RF in the y direction.

The term with coefficient b_0 returns to particles the angular momentum of RF.

The term with coefficient b_h is a force reducing the element of spin tensor related with the spatial separation of the field from the particles in the direction of h.

The term with coefficient b_y is a force reducing the element of spin tensor related with the spatial separation of RF from the particles in the y direction.

One may see from these comments that all 6 terms in the decay forces (19),(20) are necessary for the decay of all components of RF state. From the other hand, the inclusion into the decay force of other terms xc would increase the number of forces of the same direction and make the system of equations more coupled. So, little freedom of choice is left, if the force is linear in the state $\mathcal{U}$ and the coupling of equations is reduced to minimum. (19),(20) is a generalization of the corresponding decay force of the one-dimensional case [39] to the case of planar motion.

Note that the state of RF in the RHS of (19),(20) is completely retarded (all 6 terms have index ^{rr}). It is not required by causality and, in [39], both completely retarded and partly retarded forces were considered for the one-dimensional case. However, in the 3-dimensional case, only completely retarded version of F^D gives equations for $\mathcal{U}$ uncoupled with equations for differences $q_{i,j} - q_{j,i}$. To avoid a lengthy analysis of equations, where the quantities of different asymptotical behaviour are coupled, we will consider here only fully retarded decay forces.

7. Decay of RF excitations

The inhomogeneous equation (18) describes the competition of two processes: the creation of new RF distortions by the force f^A and the decay of RF distortion with the help of force F^D. The decay is expressed by the homogeneous equation

$$\mathcal{U}^0 = \mathrm{AB}\,\mathcal{U}^{(\mathrm{ret})}: \quad (21)$$

Operator A depends on coordinates and velocities of particles, depending, in their turn, on time. Operator B may depend on velocities and distances. With full account of this dependence, the analytic consideration of solutions of equation (21) is difficult even for the simplest choice of operator B.

To simplify the task, we will use synchronization (8) with time (10) and consider the estimates of the asymptotic behaviour of $\mathcal{U}$ in the rough adiabatic approximation, where change of vectors $h_i; h_j; R$ during retardation time is neglected, so that

$$ {}^{0}_{i} = {}^{0}_{j} = 1; \quad H \cdot R = 0; \quad C^r = C; ::: $$

and retardations of arguments are the same for both particles. In this approximation, system (21) is similar to system

$$ \mathcal{U}^0 = Z\mathcal{U}; \tag{22} $$

where Z is a constant matrix and which can be analysed by finding the proper values of Z. The main difference of (21) from (22) is that the state $\mathcal{U}$ in RHS of (21) has some retarded arguments. To reduce (21) formally to (22), we introduce the retardation operator E acting on elements of state

$$ E: \quad Eq = q^{\text{ret}}; \quad EQ = Q^{rr}; \quad E\mathcal{S} = \mathcal{S}^{rr}; $$

and include this operator in the elements of Z so that

$$ Z\mathcal{U} = AB\mathcal{U}^{(\text{ret})}: $$

Equation (22) has exponential solutions $\mathcal{U} = \mathcal{U}(0)\exp(t \ )$. The retardation operator acts on them as the multiplication operator:

$$ E\mathcal{U} = \exp(-t_r \ )\mathcal{U}; $$

where $t_r \geq 0$ is the retardation (in our approximation, $t_r = R^r \cdot H^r{=}|H^r| = (T + D){=}|H|$). The spectrum of proper values   for (22) can be found from the characteristic equation

$$ |Z - \ I| = 0; $$

where the elements of Z containing operator E depend on .

If all the proper values of operator matrix Z have negative real parts, force F^D will constantly diminish RF distortion $\mathcal{Q}; \mathcal{S}$ and solutions will be stable [36]. Let us write matrix Z for forces (19),(20) explicitly.

In case $f^A = 0$ and ${}^{0}_{i} = {}^{0}_{j} = 1$, (16),(17) turn into

$$ \mathcal{Q}^0 = -(F^D_{ij} + F^D_{ji}); \tag{23} $$

$$ \mathcal{S}^0 = -[R_i \wedge (F^D_{ij} - F^D_{ji}) + H \wedge \mathcal{Q}]{=}2: \tag{24} $$

The combinations of forces (19), entering these equations, in our approximation are

$$ F^D_{ij} + F^D_{ji} = a_0\bar{H}(\mathcal{Q}_H)^{rr} + a_h\bar{h}(\mathcal{Q}_h)^{rr} + b_h\bar{h}(\mathcal{S}_{Hh})^{rr} + 2a_y y(\mathcal{Q}_y)^{rr} + 2b_y y(\mathcal{S}_{Hy})^{rr}; \tag{25} $$

$$ F^D_{ij} - F^D_{ji} = a_0\bar{h}(\mathcal{Q}_H)^{rr} + 2b_0 y(\mathcal{S}_{hy})^{rr} + a_h\bar{H}(\mathcal{Q}_h)^{rr} + b_h\bar{H}(\mathcal{S}_{Hh})^{rr}: $$

Passing to scalar products with H; h; y, we get from (23),(25)

$$\mathcal{Q}_H^0 = -2\mathsf{a}_0(\mathcal{Q}_H)^{rr}; \qquad \mathcal{Q}_h^0 = 2\mathsf{a}_h(\mathcal{Q}_h)^{rr} + 2\mathsf{b}_h(\mathcal{S}_{Hh})^{rr};$$
$$\mathcal{Q}_y^0 = 2\mathsf{a}_y(\mathcal{Q}_y)^{rr} + 2\mathsf{b}_y(\mathcal{S}_{Hy})^{rr}:$$

The scalar products with $\mathrm{F}^- = \mathrm{F}_{ij}^D - \mathrm{F}_{ji}^D$ are

$$\mathrm{F}_H^- = 2\mathsf{a}_h(\mathcal{Q}_h)^{rr} + 2\mathsf{b}_h(\mathcal{S}_{Hh})^{rr}; \qquad \mathrm{F}_h^- = -2\mathsf{a}_0(\mathcal{Q}_H)^{rr};$$

$$\mathrm{F}_y^- = -2\mathsf{b}_0\mathcal{S}_{hy}^{rr}:$$

This and (24) give

$$\mathcal{S}_{hy}^0 = -\mathsf{a}_0\mathrm{R}_y(\mathrm{Q}_H)^{rr} + \mathsf{b}_0\mathrm{R}_h(\mathcal{S}_{hy})^{rr};$$

$$\mathcal{S}_{Hh}^0 = \mathsf{a}_0\mathrm{R}_H(\mathrm{Q}_H)^{rr} + \mathsf{a}_h\mathrm{R}_h(\mathrm{Q}_h)^{rr} + \mathsf{b}_h\mathrm{R}_h(\mathcal{S}_{Hh})^{rr} - \frac{\mathrm{H}^2}{2}\mathrm{Q}_h;$$

$$\mathcal{S}_{Hy}^0 = \mathsf{b}_0\mathrm{R}_H(\mathcal{S}_{hy})^{rr} + \mathsf{a}_h\mathrm{R}_y(\mathrm{Q}_h)^{rr} + \mathsf{b}_h\mathrm{R}_y(\mathcal{S}_{Hh})^{rr} - \frac{\mathrm{H}^2}{2}\mathrm{Q}_y:$$

One may note, that equations for $\mathcal{Q}_h; \mathcal{S}_{Hh}$ are fully coupled and equations for $\mathcal{Q}_y; \mathcal{S}_{Hy}$ are fully coupled. It happens because the corresponding terms in the decay forces have the same direction. The equation for $\mathcal{S}_{hy}$ remains uncoupled from the equation for $\mathcal{Q}_y$ in spite of the coincidence of the directions of the relevant terms in the force, since one term is odd to the interchange of particles, while the other term is even.

It is convenient to group the fully coupled equations together and write $\mathcal{U}$ as a column

$$\begin{pmatrix} \mathcal{Q}_H \\ \mathcal{S}_{hy} \\ \mathcal{Q}_h \\ \mathcal{S}_{Hh} \\ \mathcal{Q}_y \\ \mathcal{S}_{Hy} \end{pmatrix}:$$

Then the matrix Z corresponding to the above equations is

$$\mathrm{Z} = \begin{pmatrix} -2\mathsf{a}_0\mathrm{E} & 0 & 0 & 0 & 0 & 0 \\ -\mathsf{a}_0\mathrm{R}_y\mathrm{E} & \mathsf{b}_0\mathrm{R}_h\mathrm{E} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\mathsf{a}_h\mathrm{E} & 2\mathsf{b}_h\mathrm{E} & 0 & 0 \\ \mathsf{a}_0\mathrm{R}_H\mathrm{E} & 0 & \mathsf{a}_h\mathrm{R}_h\mathrm{E} - \frac{H^2}{2} & \mathsf{b}_h\mathrm{R}_h\mathrm{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mathsf{a}_y\mathrm{E} & 2\mathsf{b}_y\mathrm{E} \\ 0 & \mathsf{b}_0\mathrm{R}_H & \mathsf{a}_h\mathrm{R}_y\mathrm{E} & \mathsf{b}_h\mathrm{R}_y\mathrm{E} & -\frac{H^2}{2} & 0 \end{pmatrix};$$

This matrix has four diagonal blocks, above which the matrix is empty. Therefore the characteristic function $= |\mathrm{Z} - \ \mathrm{I}|$ is a product of four functions $= {}_1\,{}_2\,{}_3\,{}_4$, where

$${}_1 = - \quad - 2\mathsf{a}_0\mathrm{E};$$

$$_2 = - \quad + b_0 R_h E;$$

$$_3 = \begin{vmatrix} - \quad + 2a_h E & 2b_h E \\ a_h R_h - \frac{H^2}{2} & - \quad + b_h R_h E \end{vmatrix} = {}^2 - \quad (2a_h + b_h R_h)E + 2b_h \frac{H^2}{2} E;$$

$$_4 = \begin{vmatrix} - \quad + 2a_y E & 2b_y E \\ -\frac{H^2}{2} & - \end{vmatrix} = {}^2 - \quad 2a_y E + 2b_y \frac{H^2}{2} E;$$

and

$$E = e^{-t_r \gamma}:$$

Function ( ) has on the complex plane the zeros of all four functions $_1; : : : ; _4$. The real part of the position of the rightmost zero of the rightmost zeros of these functions determines the general asymptotical behaviour of the solution $\mathcal{U}$. If this real part is negative, the forces diminish $|\mathcal{U}|$ and make the motion of particles stable. Otherwise, they make it unstable.

By the change of variables $= t_r$, the analysis of zeros of all factors in is reduced to the analysis of two functions

$$z_1 = \quad + \quad e^{-\lambda}; \qquad z_2 = {}^2 + (\quad + \quad)e^{-\lambda};$$

where ; ; are parameters expressible through $a_0; : : : ; b_y$.

Consider function z_1. At small the rightmost root $_0$ of equation $z_1 = 0$ is real and close to $-$. At the point $= 1=e$, where the function $( ) = - \quad e^{\lambda}$ reaches the maximum ˆ, the rightmost root is $_0 = -1$ and becomes double. At greater , the root $_0$ splits into two complex roots, which move to the right with the growth of . At $= \quad =2$, roots $_0$ become purely imaginary: $_0 = \pm i \quad =2$. Therefore, the region of , where Re $_0 < 0$, and the extremal point in it are

$$0 < \quad < \quad =2; \qquad \hat{} = 1=e; \qquad _0(\hat{}) = -1:$$

Returning to functions $_1; _2$, we obtain limitations on parameters $a_0; b_0$:

$$0 < 2a_0 t_r < \quad =2; \qquad 0 < -b_0 R_h t_r < \quad =2:$$

The fastest decay rate of components $\mathcal{Q}_H; \mathcal{S}_{hy}$ is $\hat{}_0 = -1=t_r$ and is reached at

$$\hat{a}_0 = 1=(2t_r e); \qquad \hat{b}_0 = -1=(R_h t_r e):$$

Consider now the function z_3. The dependence of the rightmost root $_0$ on parameters ; is easier to analyse numerically. The calculations give the limitation

$$0 < \quad < 1:5 : : : ; \qquad 0 < \quad < f(\ );$$

where the borderline curve f looks as distorted semicircle. At small , $f^0(\ ) \approx 0:8$. Then, at about $= 1$, f reaches maximum $f_{max} \approx 0:55$. Near the other end $\approx 1:5$, the curve goes steeply to zero, $f^0 \to -\infty$ at the end point.

The fastest decay rate $\hat{\ }_0 \approx -0{:}55$ corresponds to

$$\hat{\ } \approx 0{:}5; \qquad \hat{\ } \approx 0{:}1006{:}$$

The fastest decay rate $\hat{\ }_0 \approx -0{:}55$ is slower than that obtained for equation $z_1 = 0$, and is, therefore, the fastest possible decay rate of RF distortion $\mathcal{U}$ as a whole.

The relevant limitations on parameters $a_h; b_h; a_y; b_y$ are obtained from limitations on $\ ;\ $ by using relations

$$-2(a_h + b_h R_h)t_r = \ ; \quad b_h H^2 t_r^2 = \ ;$$

$$-2a_y t_r = \ ; \quad b_y H^2 t_r^2 = \ :$$

The fastest possible decay is obtained for

$$\hat{b}_h = \frac{\hat{\ }}{H^2 t_r^2}; \quad \hat{a}_h = -\frac{\hat{\ }}{2t_r} - \frac{\hat{\ }R_h}{H^2 t_r^2};$$

$$\hat{b}_y = \frac{\hat{\ }}{H^2 t_r^2}; \quad \hat{a}_y = -\frac{\hat{\ }}{2t_r}:$$

The values $\hat{a}_0; \hat{b}_0; :::$ turn into infinity at $t_r = 0$ and at $R_h = 0$ and become large near these points, what contradicts to our assumption that accelerations are small. To remain within the region, where the adiabatic approximation and the analysis above are applicable, one has, estimating the decay rates, to consider parameters $a_0; :::$, limited everywhere, for example, the expressions

$$a_0 = \frac{1}{2e(t_r + D_0)}; \quad b_0 = -\frac{1}{e}\frac{R_h}{R_h^2 + |h^2|R_y^2 + |h^2|D_0^2}\frac{1}{t_r + D_0}; :::$$

$$b_h = \frac{\hat{\ }}{H^2(t_r + D_0)^2}; \quad a_h = -\frac{\hat{\ }}{2(t_r + D_0)} - \frac{\hat{\ }R_h}{H^2(t_r + D_0)^2};$$

$$b_y = \frac{\hat{\ }}{H^2(t_r + D_0)^2}; \quad a_y = -\frac{\hat{\ }}{2(t_r + D_0)}:$$

where D_0 is some characteristic small distance below which the decay forces are switching off. The term $|h^2|R_y^2$ in the dominator of the expression for b_0 does not let b_0 to grow near the points, where R is orthogonal to h, when R is large compared to D_0.

We may conclude, that the parameters in the expressions for the forces F^D can be chosen in such a way, that RF distortion will decay at any energies. The decay rate is limited: if time is measured in terms of the retardation time t_r, it is limited by the value $\ _0 \approx -0{:}55$. The maximal decay rate is possible everywhere, except the vicinities of the points $t_r = 0, R_h = 0$, where the adiabatic condition puts additional limitations.

8. Conditions of Elastic Scattering

The conditions of elastic scattering are specially important in the relativistic mechanics with interactions through RF since they limit the domain of consistency of the theory. Indeed, RF belonging to a pair of particles, as a mediator of interactions between these particles, should not disconnect with them. If particles, when they fly to infinity after the scattering, cease to interact with their RF and leave it in a nonzero state, it, generally, means that the used reduced description of their fields is contradictory and more detailed description of the fields is required.

The conditions of elastic scattering can be derived by a method similar to the method of the preceding section. One has only to take into account that distances between flying away particles may considerably grow during the retardation time. The equations with changeable retardation may be reduced to the equations with constant retardation parameter by a change of variables and by making the parameters of the decay force to be time-dependent.

Let us estimate the retardation time for flying away particles. At some distance from the collision zone, the trajectories of particles become close to the straight rays coming from one point. Taking this point as the origin of our coordinate frame, we may approximate the trajectories of particles by expressions

$$x_i = h_i \ _i; \quad x_j = h_j \ _j:$$

Solving then the retardation equation

$$(h_i \ _i - h_j \ _j^{ret})^2 = 0$$

for $\ _j^{ret}$ (and solving similar equation for $\ _i^{ret}$), we obtain

$$\ _j^{ret} = k \ _i; \quad \ _i^{ret} = k \ _j; \quad k = C - \sqrt{C^2 - 1} = \frac{1}{C + \sqrt{C^2 - 1}}:$$

The retardation time t_r depends on the definition of the evolution parameter t. In case of the ray trajectories, any reciprocal synchronization gives $\ _i^r = \ _j^{ret}$; $D_i = D_j$; $T_i = T_j$ and $t^{ret} = kt$, if t is proportional to . (In particular, it is true for (8) and (11) synchronizations. Besides, for the ray trajectories (10) give $t = \ _i = \ _j$.) Hence,

$$t_r = t - t^{ret} = t(1 - k):$$

Consider now the homogeneous part of one pair of coupled equations for RF components, for example, the equations for $Q_y; S_{Hy}$:

$$Q_y^0 = 2a_y Q_y(t - t_r) + 2b_y S_{Hy}(t - t_r);$$

$$S_{Hy}^0 = -\frac{H^2}{2} Q_y(t):$$

Let

$$2a_y = \tilde{a}/t_r; \quad 2b_y = \tilde{b}/(t_r)^2; \quad t = e^{\theta}.$$

Then

$$\frac{dQ_y}{d} = \frac{\tilde{a}}{(1-k)} Q_y(\; - \;) + \frac{\tilde{b}}{(1-k)t_r} S_{Hy}(\; - \;);$$

$$\frac{dS_{Hy}}{d} = -\frac{H^2}{2} t Q_y(\;).$$

The corresponding characteristic equation is

$$\begin{vmatrix} - \; + \frac{\tilde{a}}{(1-k)} E & \frac{\tilde{b}}{(1-k)t_r} E \\ -(C+1)t & - \end{vmatrix} = \; ^2 - \; \frac{\tilde{a}}{(1-k)} e^{-\gamma\mu} + \frac{\tilde{b}}{(1-k)^2}(C+1)e^{-\gamma\mu} = 0.$$

Comparing it with the equation $z_2 = 0$ of the preceding section, we may identify

$$-\tilde{a}\frac{}{1-k} = \; ; \quad \tilde{b}(C+1)\left(\frac{}{1-k}\right)^2 = \; ; \qquad _0 = \; _0 \approx -0.55;$$

what gives the decay parameters

$$a_y = -\frac{}{2}\frac{1-k}{t_r}; \quad b_y = \frac{}{2(C+1)}\left(\frac{1-k}{t_r}\right)^2.$$

In terms of time t, the decay now is not exponential, but obeys a power law:

$$(Q_y; S_{Hy}) \sim t^{\lambda/\mu}. \tag{26}$$

The consideration of other equations in the system $\mathcal{U}' = Z\mathcal{U}$ is quite similar. In all the expressions for $a_0; \ldots; b_y$, derived in the adiabatic approximation, the account of the growing separation of particles after the scattering gives the replacement of each factor $1/t_r$ by the factor $(1-k)/(\; t_r)$, and of the exponential law $e^{\lambda t/t_r}$ by the power law (26).

The after-collision decay of RF with the decay forces (19),(20) is slow at high velocities. In principle, it is sufficient to make the scattering elastic and the theory with RF consistent at arbitrary energies. However, when the decay rate is small, the terms which were neglected in the above derivation, become important and may prevent the decay of spin components of RF at high velocities. The numerical tests show that, for the fully retarded version of decay forces and for the coefficients derived above, the scattering remains elastic below the relative velocity 0.3 c. To make the scatterin elastic at higher velocities, the "less retarded" decay forces, like those considered in [39], with $Q^r; S^r$ instead of $Q^{rr}; S^{rr}$ should be used.

If the mechanics with retarded interactions is used as a model of processes with particles which scatter inelastically above some energy threshold, there is no reason to use in the model the decay forces making the scattering always elastic. The model will be more accurate below the energy threshold, if the energy dependence of the decay forces is so modified that at the threshold energy the coefficients $a_0; \ldots$ leave the region where all the roots of the characteristic equation for $\mathcal{U}' = Z\mathcal{U}$ have a negative real part.

9. Exact and Numerical Solutions

We give here just a few examples of the solutions of the equations of motion with the discussed forces.

The systems with the primary and decay forces described above permit exact solutions with circular trajectories of particles. If the center of mass of the whole system of particles and RF is placed at the origin and the particle i at t = 0 is on the x-axis, the circular solutions may be written as $(x_i)_0 = (x_j)_0 = t$; $(x_i)_3 = (x_j)_3 = 0$ and

$$(x_i)_1 = \ _i\cos(!\,t); \quad (x_j)_1 = - \ _j\cos(!\,t + \ );$$
$$(x_i)_2 = \ _i\sin(!\,t); \quad (x_j)_2 = - \ _j\sin(!\,t + \ ):$$

They are completely fixed by four parameters $\ _i;\ \ _j;\ !;\ \ .$

The family of such solutions is two-dimensional. Indeed, the fixation of the trajectories determines the primary forces and the coefficients before the scalars $Q_H; :::; S_{Hy}$ in the expressions (19),(20) for the decay forces. These scalars for the circular trajectories with constant parameters must be constant:

$$(Q_H)^0 = 0; :::; (S_{Hy})^0 = 0: \tag{27}$$

The last conditions and the RF part (3),(4) of the equations of motion give 6 equations permitting to express 6 scalars $Q_H; :::; S_{Hy}$ through the primary forces, and, hence, through the parameters $\ _i;\ \ _j;\ !;\ \ .$ So, finally, these four parameters determine the total forces F. The particle part (1),(2) of the equations of motion in case of circular solutions reduces to 6 scalar equations, two of which — $h_i \cdot F_{ij} = 0; \quad h_j \cdot F_{ji} = 0$ — are always fulfilled due to the structure of forces, another two equations

$$(F_{ij})_0 = 0; \quad (F_{ji})_0 = 0$$

are the consequences of equations (27) and are satisfied for any values of the parameters $\ _i;\ \ _j;\ !;\ \ .$ The remained two equations for the forces directed toward the origin

$$F_{ij} \cdot x_i = \ _i + m_i !^2 \ _i = (1 - \ !^2 \ _i^2) = 0;$$
$$F_{ji} \cdot x_j = \ _j + m_j !^2 \ _j = (1 - \ !^2 \ _j^2) = 0;$$

reduce the number of independent parameters from four to two. For example, one may use ! and , or the total mass $M = |P_{\mathrm{total}}|$ and the total angular momentum $J = (s_{\mathrm{total}})_{12}$ as a pair of independent parameters.

The region of $!;\ $, where the circular solutions exist, depends on the choice of forces and is limited. (In the nonrelativistic limit, $\ \to 0$, and this region becomes one dimensional. Physically, it corresponds to the vanishing of RF in the limit of small velocities due to the complete decay of all the non-energy components of RF during typical nonrelativistic times 1=!. In the absence of RF spin, the total angular momentum reduces to the angular momentum of the particles and becomes functionally dependent of the kinetic energy.)

Consider a numerical example of a relativistic circular solution. Let particles have different masses $m_i = 0{:}15$; $m_j = 0{:}85$, the principle force have form

$$F^P = c_1 \frac{h \cdot (R^r \wedge h)}{1 + (T{=}c_2)^4}; \qquad c_1 = 0{:}001; \; c_2 = 40$$

and decay force F^D have form (19),(20) with the coefficients

$$a_0 = \frac{1}{2et_r}; \quad b_0 = \frac{0{:}2}{et_r} \frac{R^{rr}_{h^{rr}}}{(R^{rr}_{h^{rr}})^2 + (h^{rr})^2 (R^{rr}_{y^{rr}})^2) + c_3^2}; \qquad c_3 = 10;$$

$$b_h = -\frac{0{:}1006}{(H^{rr})^2 (t_r)^2}; \quad a_h = \frac{1}{4t_r} - b_h R^r_{h^r};$$

$$a_y = \frac{1}{4t_r}; \quad b_y = b_h{:}$$

corresponding to the maximal decay rates of all the components of RF, except for the coefficient b_0 having additional factor 0.2, slowing down the decay of the angular momentum of RF. (Such choice of b_0 is convenient for the illustrations below.)

Let the "equilibrium" value V

In this example, the total mass of the system is greater than the sum of masses, so the particles, if they return the energy and the angular momentum, borrowed from RF, back to RF, would still have considerable kinetic energy (the lighter particle would have the velocity 0.507 c). So, one may suspect that the solution is unstable.

Db.1(y(i)-9.7g(.)5s)4.7(the)6328.r(t)-1.3(e)-7.t(r)-1.3(a)7 [illegible]

First of all, the particle systems with attractive forces, besides the ordinary scattering solutions, usually have the capturing solutions. By "capturing" we mean the process when the particles initially are far away from each other and move toward each other, but, after entering the region of interactions, instead of scattering and then flying away, they start rotating indefinitely around each other. Other words, the manifold of solutions usually has normal or strange attractors.

At the beginning of capturing process (Fig.5), the particle motion is quasi-precessing. Then the trajectories either turn gradually into a precessing solution, or converge to a circular one (to the solution on Fig.1 in case of Fig.5), or to a non-periodic solution (strange attractor). Since the region of existence of the circular solutions is two-dimensional, the limiting circular solutions are not at all rare (as they are in the Hamiltonian theory, where their family is usually one-dimensional in $\mathbf{M}; \mathrm{J}$ plane), but are quite common. For some interactions, most of the region of initial parameters (energies and angular momenta), leading to capturing, is covered by the region of the circular solutions.

Besides the capturing solutions and of the simple scattering solutions, there is often a region of quasi-capturing solutions, where colliding particles start orbiting, then orbits become close to quasi-precessing solutions slowly changing their parameters. When these parameters reach some critical values, the motion becomes unstable, and the particles fly away. Such solutions can be considered as classical analogies of the quantum resonant states of the elementary particle physics.

Another interesting feature of the motion with retarded interactions is the non-trivial motion of the center of mass of two particles, which may be orbiting around the center of mass of the total system at considerable distance.

with a continuous medium. It may reproduce qualitatively even the existence of the energy threshold between the elastic and inelastic scattering, which is usually considered as a purely quantum-field phenomenon related to particle production. These features of the theory with reduced fields make this theory very promising for the construction of the classical models of various processes with elementary particles which before could be considered only in the frame of the relativistic quantum field theory.

The equations of motion with RFs permit to take the electromagnetic interactions into account exactly by using the Lienard-Wiechert forces with the radiation friction term [41]

$$\mathrm{F}_i^{\mathrm{rf}} = \frac{2}{3}\mathrm{e}^2\mathrm{h}_i \cdot (\mathrm{h}_i \wedge \ddot{\mathrm{h}}_i);$$

where $\ddot{\mathrm{h}}$ (as well as $\dot{\mathrm{h}}$) should be understood and numerically realized as a limit from the left to exclude unphysical growing solutions. Then the state U of the reduced electromagnetic field (in case of several particles, the sum of such states) will give the correct values of all the energy-momenta of the radiated field without actual calculating of the outgoing waves what is specially convenient in case of several charged particles [42].

It is worth to comment briefly the way of getting the orbital motion in the theory with RFs and in earlier MT relativistic models for a pair of electrically charged particles. Such models use the combinations of retarded and advanced interactions, introduced either in a symmetrical (e.g. [2]), or in an asymmetrical (e.g. [43]) fashion. In both cases, inclusion of advanced interactions eliminates the radiation losses (what, of course, is a distortion of the real electromagnetic interactions though excused usually by the references to the action-at-a-distance version of the electrodynamics [44]). The actual justification of the advanced interactions, violating the causality, is the desire to obtain closed (or precessing) orbits and to come to the Hamiltonian or Lagrangian description.

In the present approach, the RF-dependent decay forces, if added to the Lienard-Wiechert forces, are able to reach the same goal, i.e. to prevent radiation losses and produce closed (or precessing) orbits, at smaller cost, without violating the causality. With respect to the particle motion, the decay forces are similar to the forces from the fields in accelerating cavities of storage rings compensating the radiation losses of charged particles.

The peculiar features of the orbital solutions classified in [43] (the solutions with the sign of $\mathrm{J}_{\mathrm{total}}$ opposite to the sign of $\mathrm{J}_{\mathrm{particles}}$, the sideway positions of the center of mass of particles) are easily reproducible in the theory with RFs. The variety of solutions in the new theory is, in fact, wider, than in the older models, and contains strange attractors and capturing solutions.

The reduced description of the mediating field and the forces, considered in this paper, are the simplest ones. One may take into account some excitations of the mediating field, adding to 10 values U describing the state of RF, other functionals of the trajectories and making the forces to depend on these functionals. The freedom here is very large. However, these additional functionals, if classified according some "mean retardation" of the parts of the trajectories used in the functionals, will lie between the echo forces, or the principle forces, which depend on the least retarded points of the trajectories, compatible with causality, and the decay forces, which depend on functionals U dependent on old parts of the trajectories up to $-\infty$. So, the combination of forces $F^E; F^P; F^D$ interpolates in some sense the whole range of differently retarded forces.

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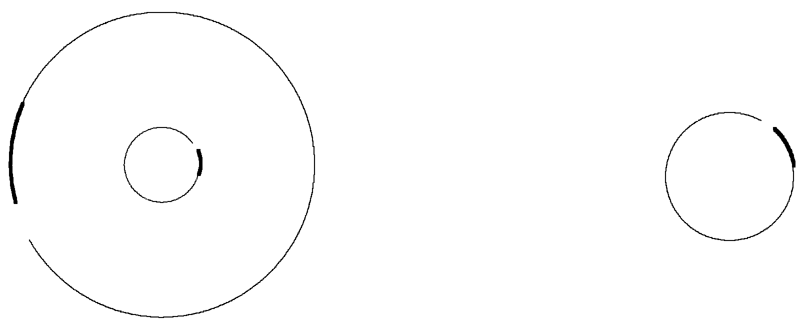
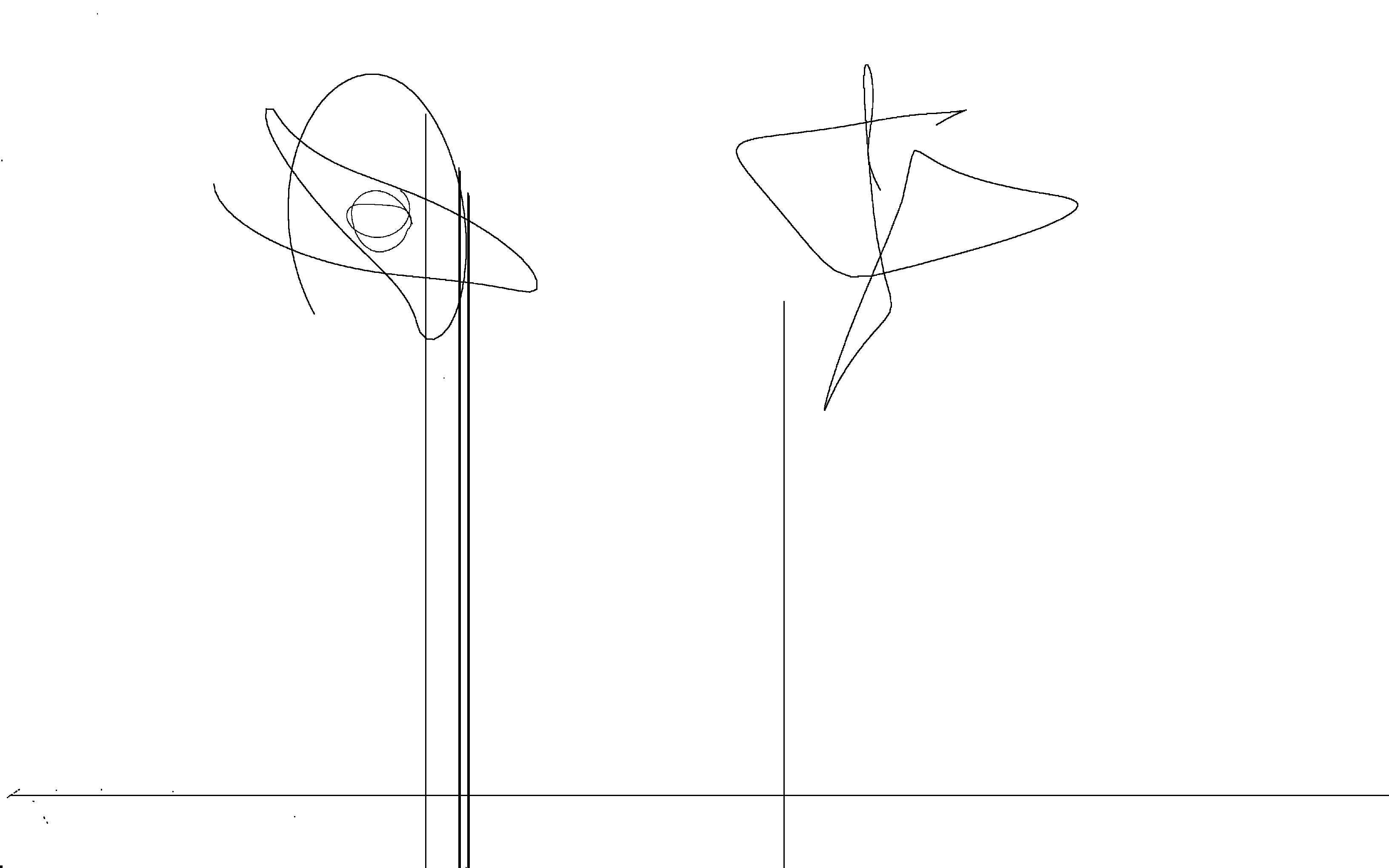


Figure 1: The circular solution (J $= 4{:}335$).



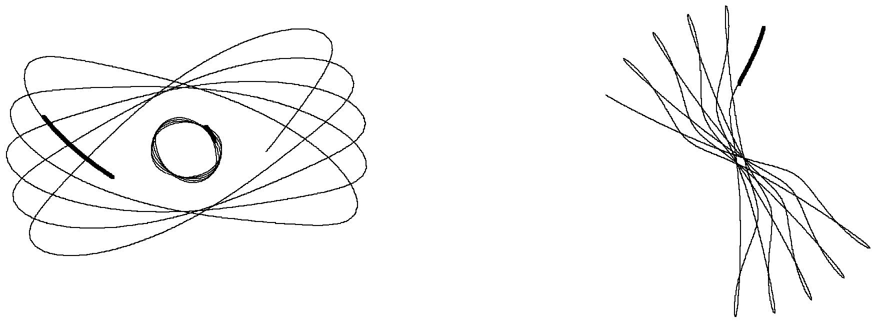


Figure 3: The precessing solution (J $= -4$).

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