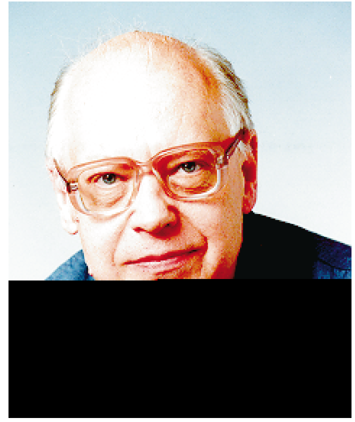
CANONICAL COMMUTATION RELATIONS — HISTORICAL ASPECT AND NEW RESULTS



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1. Introduction

Canonical commutation relations (CCR) are the algebraic structure (Heisenberg algebra) at the basis of quantum mechanics (QM) and quantum field theory (QFT).

They are of great interest not only from a physical point of view, but also as a basic mathematical structure.

The Heisenberg algebra is the Lie algebra generated by the so-called canonical variables $q_i, p_i, i = 1, \cdots n$, satisfying the following relations:

$$[p_i, q_j] = -i\delta_{ij}. \tag{1}$$

Foundations of QM rely on the analysis of the representations of such an algebra. In the case of infinite numbers of degrees of freedom ($n = \infty$) CCR play an important role in the construction of QFT.

We start with the case of finite numbers of degrees of freedom. It is sufficient to describe the simplest case $n = 1$ as the extension to arbitrary finite numbers of degrees of freedom being straightforward. So we restrict our attention to the relation:

$$[p, q] = -i. \tag{2}$$

2. The first results

In the beginning two realizations of CCR have been obtained: the Heisenberg [1] and the Schrödinger [2] representations.

Heisenberg constructed the representation in the form of infinite matrices:

$$q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ \dots & & & & \\ & & & & \end{pmatrix},$$

$$p = \frac{-i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & \dots \\ \dots & & & & \\ & & & & \end{pmatrix}. \tag{3}$$

The Schrödinger representation of CCR is the realization of CCR in the space $L_2(-\infty,\infty)$, so in the space of all square-integrable functions $\varphi(q) : \int_{-\infty}^{\infty} |\varphi(q)|^2 \, dq < \infty$, where the integral is regarded as Lebesgue one. q is a multiplication operator and p is a differentiation one: $p = -i\frac{d}{dq}$.

Strictly speaking CCR are defined on some dense domain D of $L_2(-\infty,\infty)$ as q and p are unbounded operators in this space. Let's recall that in closed spaces (e.g. in Hilbert space) unbounded operator can't be defined in all space, but on some dense domain only (this and other results of functional analysis used in this report can be found in [3, 4]). This point is very important. In fact, as it has been proved by Wintner [5], no realization of CCR with both bounded operators q and p exists. Really, even more strong assertion is true [6] (see also [7, page 2]):

Proposition 1. *In an arbitrary normalized space identical operator I cannot be expressed as commutator of two bounded operators.*

Proof. Suppose that

$$[A, B] = I, \tag{4}$$

where A and B both are bounded.
From eq.(4) it follows directly that

$$(n+1)B^n = AB^{n+1} - B^{n+1}A; \qquad n \in \mathbf{Z}^+. \tag{5}$$

Thus

$$(n+1)||B^n|| \leq 2||A|| \, ||B^{n+1}|| \leq 2||A|| \, ||B|| \, ||B^n||,$$

and $B^n = 0$ at sufficiently large n. As it follows from eq.(5), the last condition implies that $B^{n-1} = 0$ as well, and so also $B^{n-2}, \dots B = 0$.

Let's point out that this Proposition is valid not only for Hilbert space, but for more general spaces as well, e.g. for Krein space (see the definition in Section 7).

At first sight the Heisenberg representation and the Schrödinger one are quite different. But really they are unitary equivalent, i.e. if p_1, q_1 form a Schrödinger representation of CCR and p_2, q_2 form a Heisenberg one, then there exists unitary operator V such that

$$p_2 = V p_1 V^+ , \qquad q_2 = V q_1 V^+ . \tag{6}$$

In fact, we can choose the basis in $L_2(-\infty, \infty)$ in such a way that matrix representations of operators q and $-id/dq$ in this space are given by matrices (3). Such basis is formed by functions $exp(-q^2/2)H_n(q)/\pi^{1/4}\sqrt{2^n n!}$, where $H_n(q)$ are Hermit polynomials. The necessary result follows directly from well known recurrent properties of Hermit polynomials. In other words Heisenberg representation is the same as Schrödinger one, but in space l_2 - the space of all infinite converges sequences $x = (x_1, x_2, \ldots x_n, \ldots)$, $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. It is well known that space $L_2(-\infty, \infty)$ is isomorphic to the space l_2. Besides, if p, q form a representation of CCR then $p' = VpV^+$ and $q' = VqV^+$ also form a representation of CCR, V is any unitary operator.

Very important step in history of CCR is a determination of class of representations of CCR which are unitary equivalent to Schrödinger one. Below we call such representations as regular. Further on we consider this point in detail. Now let's bring a simple example of the representation which is non-unitary equivalent to the Schrödinger one. This example is a representation of CCR by operators of multiplication and differentiation (just as in Schrödinger one), but in space $L_2(0, 1)$. Dense domain D, where CCR are fulfilled is a domain defined by the conditions $\varphi(1) = exp\,(i\theta)\varphi(0)$. Evidently this representation is not unitary equivalent to Schrödinger one as operator q is a bounded operator in $L_2(0, 1)$ and unbounded one in $L_2(-\infty, \infty)$. Unboundedness of operators p and q (or at least of one of them) gives rise nontrivial mathematical problems in investigation of CCR.

In fact, two distinguish possibilities exist:

1. to consider accurately the delicate domains questions in the standard form of CCR;
2. to substitute CCR by analogous relations with bounded operators.

Both these possibilities were realized. Let's start with the second one.

3. Weyl form of CCR

If we forget for a moment that p and q are unbounded, then we can carry out the simple calculations given below. From eq.(2) it follows directly that

$$[p, q^n] = inq^{n-1} = -i(q^n)' . \tag{7}$$

As $exp\,(isq) = \sum_{n=0}^{\infty} \frac{(isq)^n}{n!}, \quad s \in \mathbb{R}$, then

$$[p, e^{isq}] = se^{isq} \tag{8}$$

or

$$e^{-isq} p e^{isq} = (p + sI), \tag{8a}$$

where I is an identical operator.
Eq. (8a) implies that

$$e^{-isq} p^n e^{isq} = (p + sI)^n \,. \tag{8b}$$

So for function $exp\,(itp), t \in \mathbb{R}$ we have finally:

$$exp\,(itp) exp\,(isq) = exp\,(its) exp\,(isq) exp\,(itp) \tag{9}$$

This relation is CCR in a Weyl form [8] (briefly Weyl relations). Let's stress that this derivation is not rigorous as, in general, $exp\,(isq)$ and $exp\,(itp)$ can't be represented by Tailor's series as q and p are not bounded (see e.g. [3, 4]). So at this stage we can consider (9) only as relations independent from (2). The connection between Weyl and standard form of CCR has been established in series of papers (see below).

The precise meaning of eq.(9) is the existence of two unitary groups: $V(s) \equiv exp\,(isq)$ and $U(t) \equiv exp\,(itp), \quad s, t \in \mathbb{R}$, which satisfy eq.(9). If the generators q and p are self-adjoint operators, then these groups are strongly continuous in accordance with Stone theorem [3, 4]. Such representations of CCR in Weyl form is said to be regular (example of non regular representation is given in [9, page 7]).

Let's point out that operators q and $p = -id/dq$, which form Schrödinger representation in $L_2(-\infty, \infty)$, generate the following unitary groups: $V(s)f(q) = exp\,(isq)f(q), \quad U(t)f(q) = f(q + t), \quad f(q) \in L_2(-\infty, \infty)$. It is easy to show that these groups satisfy the Weyl relations.

Very important step in the classification of regular representations has been made by Von Neumann [10].

4. Von Neumann's uniqueness theorem

Below for simplicity we consider irreducible representations only. (In fact, any regular representation of CCR in Hilbert space is a direct sum of regular irreducible representations (see [7] or [9] for details).

Von Neumann's uniqueness theorem *If on a Hilbert space H self-adjoint operators q and p are generators of the unitary groups $V(s) = exp\,(isq)$ and $U(t) = exp\,(itp)$, satisfying the Weyl relations (9), then q and p satisfy a regular representation of CCR (see eq.(2)), i.e. representation, which is unitary equivalent to Schrödinger one.* (For proof see e.g. [7, page 65] or [9, page 5]).

Von Neumann's theorem remains unsettled two important questions:
1. What kind of conditions (without relying to Weyl relations) have to satisfy p and q in order that the corresponding representation of CCR be regular?
2. In what case a regular representation of CCR in usual form gives rise the corresponding Weyl relations and vice versa.

The first problem has been solved in papers of Rellich [11] and Dixmier [12], the second - in papers of Foias, Gehér and Sz-Nagy [13] (the generalization of the last result has been done by Kato [14]).

5. Regularity conditions

Let's formulate necessary and sufficient conditions for regularity of a representation of CCR (Rellich-Dixmier conditions).

Proposition 2. *p and q form regular representation of CCR if:*
1. there exists dense domain $D \in D_p \cap D_q$ invariant under the action of p and q such that CCR hold on D;
2. Operator $p^2 + q^2$ is essentially self-adjoint on D.

The proof see in [7].

Let's point out the importance of condition (2). Fuglede [15] has constructed interesting example of non-regular representation of CCR, where condition (1) is fulfilled and moreover p and q are essentially self-adjoint on D.

We see that rather general conditions on p and q determine regular representation of CCR. So it is natural to admit Rellich-Dixmier conditions as a definition of regularity of representation of CCR. From this point of view Schrödinger representation is regular as it satisfies Rellich-Dixmier conditions.

The conditions of regularity can be formulated in a different way. To this end let's construct "annihilation" a and "creation" a^+ operators:

$$a = \frac{q + ip}{\sqrt{2}}; \quad a^+ = \frac{q - ip}{\sqrt{2}}. \tag{10}$$

It is evident that a and a^+ are conjugated operators as p and q are self-adjoint.

From (2) and (10) it follows immediately that

$$[a, a^+] = 1. \tag{11}$$

Strictly speaking the unboundedness of operators p and q leads to necessity of a careful check of the equivalence of eqs. (2) and (11), for details see [7, page 70]).

It is easy to check that in the Schrödinger representation there exists "vacuum" vector ψ_0 satisfying the condition

$$a\psi_0 = 0. \tag{12}$$

Actually, $a = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$ and $\psi_0 = const\, exp\,(-\frac{q^2}{2})$.

All other eigenvectors of operator $N = a^+a$ in the Schrödinger representation are obtained by the action of operators $(a^+)^n$ on ψ_0 and the corresponding space is a span of these vectors. So ψ_0 is a cyclic vector of the algebra in question. This property of ψ_0 and condition (12) were regarded in paper of Dubin and Hennings as definition of "s-class" of representation [16]. Such definition is equivalent to our definition of regularity.

Under sufficiently general conditions Tillmann [17] and Putnam [18] have proved that operator N has district spectrum and this spectrum is $\mathbb{N}$. Precisely it is necessary to suppose that a is a closed, densely defined operator, $D(aa^+) = D(a^+a)$ and relation (11) is hold on this set. Then N is a self-adjoint operator and spectrum $N = \mathbb{N}$. Thus, vacuum vector exists in such a representation. (For proof see [7, page 68].) All mentioned conditions of regularity are equivalent as all they determine the same class of representations of CCR. Let's stress that specifically these representations are relevant to QM.

6. The connection between regular representations of CCR in standard and Weyl form

As was already mentioned the advantage of Weyl form of CCR (9) relates with boundedness of the unitary operators $V(s)$ and $U(t)$. But in QM or QFT it is more convenient to work directly with operators a and a^+.

In paper of Foias, Gehér and Sz-Nagy [13] the conditions on p and q under which corresponding groups $U(t)$ and $V(s)$ satisfy Weyl relations (9) were found. It means that in accordance with Von Neumann's theorem (Section 4) p and q belong to the regular representation of CCR. It is interesting that formally these conditions are quite different from Rellich-Dixmier conditions.

Proposition 3. *Representation of CCR is regular if p and q are self-adjoint operators and there exists a linear set D, contained in D_{pq-qp}, such that*
1. $(p + iI)(q + iI)D$ or $(q + iI)(p + iI)D$ be dense in H;
2. CCR are hold on D.

In accordance with the results of Foias, Gehér and Sz-Nagy any regular representation of CCR gives rise regular representation of Weyl relations. The proof also see in [7].

Besides it has been proved in paper [13] that if groups $V(s)$ and $U(t)$ satisfy Weyl relations with necessary continuity properties, then the generators of these groups belong to a regular representation of CCR. So, if groups $U(t)$ and $V(s)$ are strongly continuous, then their generators necessarily form regular representation of CCR and thus these continuity properties define regular representation of CCR. Let's stress that for regular representations Weyl relations are equivalent to standard CCR. In paper of Kato [14] similar results have been obtained for the Banach space.

7. CCR in an indefinite metric space

All previous results concern CCR in Hilbert space H, but in gauge quantum field theory (GQFT) more general spaces, namely the spaces where inner product $< x, x >$ may not be positive, are widely used, especially under rigorous treatment (see [19, 20, 21] and references herein, properties of indefinite metric spaces are described in [22, 23]). The quantization in indefinite metric spaces has been proposed by Dirac [24] and Pauli [25]. Such spaces crucially enter in the local formulation of GQFT's, where locality and covariance of the gauge fields are incompatible with positivity of the inner product [19, 20, 21].

Regular representations in an indefinite metric space have been described in paper of Mnatsakanova, Morchio, Strocchi and Vernov [26], (see also [27]). On the basis of these papers here we consider regular representations in Hilbert and in Krein spaces in unique way (in fact all results remain valid in any weakly completed space). Let's recall that Krein space K can be represented as a direct sum of Hilbert and Anti-Hilbert spaces [22, 23]:

$$K = K^+ \oplus +K^-, \qquad K^+ \perp K^- . \tag{13}$$

So any $x \in K$ is:

$$x = x^+ + x^-, \quad x^\pm \in K^\pm \quad x^+ \perp x^- . \tag{14}$$

In Krein space Hilbert scalar product (x, y) can be introduced as well as indefinite one $< x, y >$, namely

$$(x, y) =< x^+, y^+ > - < x^-, y^- > . \tag{15}$$

As $K^\pm$ are completed spaces, K is completed as well and Krein norm $||x||$ can be defined as $(x, x)^{\frac{1}{2}}$.

If we introduce operator J (metric operator):

$$J(x^+ + x^-) = x^+ - x^-,$$

then it is easy to check that

$$(x, y) =< x, Jy >, \qquad < x, y >= (x, Jy). \tag{16}$$

Let's also point out that K is a non-degenerate space, that is: if $x \perp K$, then $x = 0$.

The first step in studying CCR in Krein space is a suitable notion of regularity (for simplicity we use slightly less general form than necessary). We define representation as regular if:

1. There exists dense domain D stable under the action of a and a^+ such that CCR are fulfilled on D;
2. There exists operator $U(s), s \in \mathbb{R}$, satisfying the following conditions on D:

$$U(s)D \in D\,; \tag{17}$$

$$U(s)U(t) = U(s+t)\,; \tag{18}$$

3.

$$\lim_{\to 0} \frac{U(\Delta) - I}{\Delta}\psi = iN\psi\,. \tag{19}$$

Here "lim" means the strong limit, i.e. the limit by Krein norm $\|\cdot\|$.

From eq.(2) it follows that on D:

$$aN = (N+1)a,$$

$$\tag{20}$$

$$a^+N = (N-1)a^+\,.$$

Conditions (17) and (19) imply that

$$\frac{d}{ds}U(s)\psi = iNU(s)\psi, \qquad \forall s \in \mathbb{R} \quad \text{and} \quad \psi \in D\,. \tag{21}$$

In accordance with (20):

$$\frac{d}{ds}\left(U(s)aU(-s)\right) = iU(s)[a^+a, a]U(-s) = -iU(s)aU(-s)\,. \tag{22}$$

So

$$U(s)aU(-s) = e^{is}a; \qquad U(s)a^*U(-s) = e^{-is}a^* \quad \text{on D.} \tag{23}$$

Owing to eq.(23)

$$[U(2\pi n), a] = [U(2\pi n), a^+] = 0; \qquad n \in \mathbf{Z}\,. \tag{24}$$

Now let's introduce the proper notion of irreducibility. As it is well known the concept of irreducibility is delicate in the case of algebras of unbounded operators (detailed analysis of this problem for the algebra in question has been done in [26]).

Here we say that representation is irreducible if any closed operator which leaves D invariant and commutes with operators of the algebra, i.e. with a and a^+, is a multiple of the identity. One can show [26] that under this definition of irreducibility K does not contain any closed subspace invariant under the algebra in question.

Thus according to eq.(24)

$$U(2\pi) = e^{2\pi i\theta}I; \qquad 0 \le \theta \le 1 \tag{25}$$

if representation in question is irreducible. Below we consider such representations.

Proposition 4. *If there exists operator $U(s)$ satisfying the conditions (17) - (19), then operator N has an eigenvector :*

$$N\psi_\lambda = \lambda\psi_\lambda\,. \tag{26}$$

Proof. Let's introduce operator $W^k(s) = \int_0^s V(s)e^{-iks}\psi ds, \quad k \in \mathbf{Z}$, where $V(s) = U(s)e^{-is\theta}$.

In accordance with (25) $V(2\pi) = I$.

The existence of this operator has been proved in [27].

Let's prove that $W^k(2\pi)\psi$ is an eigenvector of $V(s)$ at arbitrary s.

Indeed,

$$V(s)W^k(2\pi)\psi = \int\limits_0^{2\pi} V(s)e^{-iks^0}V(s')\psi ds' = \int\limits_0^{2\pi} V(s+s')e^{-iks^0}\psi ds' =$$

$$= e^{iks}\int\limits_s^{2\pi+s} V(s')e^{-iks^0}\psi ds' = e^{iks}W^k(2\pi)\psi. \tag{27}$$

In the last step we use eq.(25). So according to (27)

$$(V(s) - e^{iks})\psi_k = 0; \quad \psi_k \equiv W^k(2\pi)\psi. \tag{28}$$

Eq.(28) implies that $\lim\limits_{\to 0}\frac{U(\ )-I}{}\psi_k$ exists, so owing to (19) $\lim\limits_{\to 0}\frac{U(\ )-I}{}\psi_k = iN\psi_k$. Finally

$$N\psi_k = (k+\theta)\psi_k. \tag{26$'$}$$

It can be easily shown that owing to non-degeneracy of K, there exists such k_0 that $\psi_{k_0} \neq 0$. Indeed, in the opposite case $\forall\phi$

$$< \phi, W^k(2\pi)\psi >= \int\limits_0^{2\pi} < \phi, V(s)\psi > e^{-iks}ds = 0 \quad \forall k.$$

Function $T(s) \equiv < \phi, V(s)\psi >$ is a periodic function, since $V(2\pi) = I$. Conditions $\int_0^{2\pi} e^{-iks}T(s)ds = 0$ imply $T(s) \equiv 0$. So $< \phi, \psi >= 0 \quad \forall\phi$ in contradiction with the non-degeneracy of K. Eq.(26) is proved.

Proposition 5. *If ψ_λ satisfies eq.(26) and eq.(2) holds, then $Sp\,N$ is discrete and vectors $a^n\psi_\lambda$ and $(a^+)^n\psi_\lambda$ exhaust the set of eigenvectors of N.*

Proof. As $a^+a = N, aa^+ = N+1$, and $D(N) \subset D(a) \cap D(a^+)$,then there exist vectors $a\psi_\lambda$ and $a^+\psi_\lambda$. According to eq.(20) $aN\psi_\lambda = \lambda a\psi_\lambda = (N-1)a\psi_\lambda$.

Thus

$$N\psi_{\lambda-1} = (\lambda-1)\psi_{\lambda-1}, \quad \psi_{\lambda-1} = a\psi_\lambda. \tag{29}$$

Similarly

$$N\psi_{\lambda+1} = (\lambda+1)\psi_{\lambda+1}, \quad \psi_{\lambda+1} = a^+\psi_\lambda. \tag{30}$$

Continuing this process we obtain:

$$N\psi_{\lambda-n} = (\lambda-n)\psi_{\lambda-n}, \quad \psi_{\lambda-n} = a^n\psi_\lambda, \tag{31}$$

$$N\psi_{\lambda+n} = (\lambda+n)\psi_{\lambda+n}, \quad \psi_{\lambda+n} = (a^+)^n\psi_\lambda. \tag{32}$$

In order to prove that this set of vectors exhausts the whole set of eigenvectors of N it is sufficient to notice that

$$M(a^n, (a^+)^m)\psi_\lambda \sim \psi_{\lambda+m-n},$$

where $M(a^n, (a^+)^m)$ is an arbitrary monomial of n operators a and m operators a^+. The proof is completed.

Remark 1. Let's point out that Propositions 4 and 5 remain valid for more general algebras then CCR. In fact, only what necessary for their proof are the conditions (17) - (19) and (20). If algebra under consideration is algebra of CCR, then eqs.(20) follow from eq.(2). But eqs.(20) are more general then CCR. Indeed, we can consider an algebra of operators a and a^+, defined by eqs.(20) and the following conditions:

$$a^+a = \varphi(N); \qquad aa^+ = \varphi(N+1), \tag{33}$$

where $\varphi(N)$ is some nonsingular function. CCR is the simplest case of such an algebra: $\varphi(N) = N$.

Other well-known algebra of this type used in QM is the algebra of q-deformed commutators:

$$aa^+ - qa^+a = q^{-N} \qquad q \in \mathbb{R}. \tag{34}$$

It is easy to see that for this algebra

$$a^+a = \frac{Cq^N - q^{-N}}{q - q^{-1}} \qquad C \in \mathbb{R}; \quad q \neq -1,$$

$$a^+a = (N+C)e^{i\pi(N+1)} \qquad C \in \mathbb{R}; \quad q = -1.$$

As now conditions (20) are basic conditions which define algebra in question we have to give the precise meaning of them. We assume that if for some vector ψ vector $aN\psi$ exists, then vector $N(a-1)\psi$ exists as well and vice versa. According to the first of eqs.(20) these vectors are equal.

Now let's define K_0 - the space of all finite sequences $\sum C_k\psi_{\lambda+k}$. It is evident that $K \supseteq K_0$.

Let's point out that for a given K_0 there exist different completions $\bar{K}_0$, corresponding to different Hilbert structure on K_0. But we can consider representations of the algebra in question in all these spaces as equivalent if they coincide on dense domain D_0 (see discussion of the similar problem in [26, page 13]).

There exists a distinguished closure $\bar{K}_0$ (distinguished Krein structure on K_0, below we admit that K coincides with this closure of K_0), which can be constructed by the following way.

Let's introduce the set of normalized eigenvectors of N:

$$\tilde{\psi}_{\lambda+k} = \frac{\psi_{\lambda+k}}{| < \psi_{\lambda+k}, \psi_{\lambda+k} > |^{\frac{1}{2}}}.$$

We exclude the possibility $< \psi_{\lambda+k_0}, \psi_{\lambda+k_0} >= 0$ at some k_0. We can do it as in the opposite case (see below eqs.(36) and (38)) $< \psi_{\lambda+k}, \psi_{\lambda+k} >= 0 \forall k$ in contradiction with non-degeneracy of space K.

Dividing the whole set of normalized eigenvectors on the set of positive e_k^+ and negative e_k^- vectors, $< e_k^\pm, e_k^\pm >= \pm 1$ (see below eqs.(37) and (39)), we see that

$$K_0 = K_0^+ \oplus K_0^-,$$

where $K_0^\pm$ is a space of all finite sequences $\sum C_k^\pm e_k^\pm$.

If

$$x^\pm = \sum_0^\infty C_k^\pm e_k^\pm,$$

then $x^\pm$ can belong to K only if $|< x^\pm, x^\pm >| < \infty$, so if

$$\sum_0^\infty |C_k^\pm|^2 < \infty. \tag{35}$$

We construct $\bar{K}_0^\pm$ by using intrinsic scalar product $< \cdot, \cdot >$ and define

$$\bar{K}_0 = \bar{K}_0^+ \oplus \bar{K}_0^-.$$

In other words we introduce the following Hilbert product $(\cdot, \cdot)$ on K_0:

$$(x, y) =< x^+, y^+ > - < x^-, y^- >,$$

if

$$x = x^+ + x^-, y = y^+ + y^-; \qquad x^\pm, y^\pm \in K_0^\pm$$

and complete K_0 in accordance with this Hilbert structure on K_0.

For such a choice of completion if $x = \sum C_k \tilde{\psi}_{\lambda+k}$, then

$$x \in \bar{K}_0 \qquad \text{iff} \qquad x = \sum_{-\infty}^{+\infty} C_k \tilde{\psi}_{\lambda+k}; \qquad |C_k|^2 < \infty. \tag{35'}$$

It is easy to see that in this space eq.(2) is valid for x if $\sum\limits_{-\infty}^{+\infty} |C_k|^2 k^2 < \infty$. So CCR is fulfilled for every $x \in D(N)$. The domain D is a subset of $D(N)$, stable under the action of operators a and a^+ and can be taken as all finite and rapidly convergent sequences of $\tilde{\psi}_{\lambda+k}$: $|k|^n C_k \to 0$ at all n.

In the Hilbert space condition (35′) turns to a standard one.

8. Classes of regular representations of CCR

Let's describe all (up to unitary equivalent) regular (irreducible) representations of CCR in Krein space.

To do this let's notice that

$$< \psi_{\lambda+n}, \psi_{\lambda+n} >=< \psi_{\lambda+n-1}, aa^{+}\psi_{\lambda+n-1} >= (\lambda+n) < \psi_{\lambda+n-1}, \psi_{\lambda+n-1} > . \tag{36}$$

So

$$< \psi_{\lambda+n}, \psi_{\lambda+n} >= (\lambda+n)(\lambda+n-1)\dots(\lambda+1) < \psi_{\lambda}, \psi_{\lambda} > . \tag{37}$$

Similarly

$$< \psi_{\lambda-n}, \psi_{\lambda-n} >= (\lambda-n+1) < \psi_{\lambda-n+1}, \psi_{\lambda-n+1} > \tag{38}$$

and

$$< \psi_{\lambda-n}, \psi_{\lambda-n} >= (\lambda-n+1)(\lambda-n+2)\dots\lambda < \psi_{\lambda}, \psi_{\lambda} > . \tag{39}$$

Without lost of generality we can always set $< \psi_{\lambda}, \psi_{\lambda} >= \pm 1$. Below we put $< \psi_{\lambda}, \psi_{\lambda} >= 1$, as another case differs in evident manner.

Evidently $\lambda \in \mathbb{R}$ as $< a\psi_{\lambda}, a\psi_{\lambda} >\in \mathbb{R}$. From eqs.(36) and (38) it follows that if $< \psi_{\lambda+k_0}, \psi_{\lambda+k_0} >= 0$ at some k_0, then all $< \psi_{\lambda+k}, \psi_{\lambda+k} >= 0$.

There exist three different type of $Sp\,N$:

1. $Sp\,N$ is bounded from below. In accordance with eqs. (36) and (39) in this case $\lambda = n$. We can always put $\lambda = 0$, then $a\psi_0 = 0$ *(Fock representation)*. It is easy to see that this case is the usual Hilbert space case, $Sp\,N = \mathbb{N}$.

2. $Sp\,N$ is bounded from above. In this case $\lambda \in \mathbb{Z}_-$ and we can put $\lambda = -1$. Evidently $a^{+}\psi_{-1} = 0$*(Anti-Fock representation)*. So $Sp\,N = \mathbb{Z}_-$. Metric is indefinite

$$< \tilde{\psi}_{-n}, \tilde{\psi}_{-n} >= (-1)^{n+1}. \tag{40}$$

We recall that

$$\tilde{\psi}_{\lambda+k} = \frac{\psi_{\lambda+k}}{| < \psi_{\lambda+k}, \psi_{\lambda+k} > |^{\frac{1}{2}}}.$$

Hilbert structure on K can be defined by metric operator J:

$$J\tilde{\psi}_{-1-n} = (-1)^{n}\tilde{\psi}_{-1-n}. \tag{41}$$

3. $Sp\,N$ is unbounded *(Dirac sea case)*. Here we can always choose λ in such way that $0 < \lambda < 1$, so $\lambda = \theta$ (see eq.(25)). According to eqs.(36) and (39)

$$< \tilde{\psi}_{\lambda+k}, \tilde{\psi}_{\lambda+k} >= (-1)^{\frac{|k|-k}{2}}. \tag{42}$$

In this case we can define:

$$J\tilde{\psi}_{\lambda+k} = (-1)^{\frac{|k|-k}{2}}\tilde{\psi}_{\lambda+k}. \tag{43}$$

It is easy to see that in all cases

$$[J, N] = 0\,. \tag{44}$$

As all $< \psi_{\lambda+k}, \psi_{\lambda+k} >$ are uniquely determined if $Sp\, N$ is fixed so all representations which correspond to given $Sp\, N$ are unitary equivalent. Indeed, if a_1, a_1^+ and a_2, a_2^+ are regular representations of CCR in spaces K_1 and K_2 respectively and $\lambda_1 = \lambda_2$, then the correspondence between K_1 and K_2 can be realized by operator V:

$$x_2 = V x_1, \quad \text{where} \quad x_i = \sum_k c_k \psi^i_{\lambda+k}; \quad i = 1, 2.$$

As $\psi^i_{\lambda+k}$ form basis in K_i, then

$$< x_1, y_1 > = < x_2, y_2 > \quad \forall x_i, y_i,$$

if

$$x_2 = V x_1, \qquad y_2 = V y_1.$$

Thus V is an isometric (really unitary) operator as correspondence between K_1 and K_2 is one-to-one. The equality $a_2 = V a_1 V^+$ immediately follows from this correspondence as $a_2 x_2 = V a_1 x_1$ if $x_2 = V x_1$.

Remark 2. If algebra under consideration is defined by conditions (33) then eqs.(36)-(39) are changed by the evident way:

$$< \psi_{\lambda+n}, \psi_{\lambda+n} > = \varphi(\lambda+n) < \psi_{\lambda+n-1}, \psi_{\lambda+n-1} >, \tag{36$'$}$$

$$< \psi_{\lambda+n}, \psi_{\lambda+n} > = \varphi(\lambda+1)\varphi(\lambda+2)\cdots\varphi(\lambda+n) < \psi_\lambda, \psi_\lambda >, \tag{37$'$}$$

$$< \psi_{\lambda-n}, \psi_{\lambda-n} > = \varphi(\lambda-n+1) < \psi_{\lambda-n+1}, \psi_{\lambda-n+1} >, \tag{38$'$}$$

$$< \psi_{\lambda-n}, \psi_{\lambda-n} > = \varphi(\lambda)\varphi(\lambda-1)\cdots\varphi(\lambda-n+1) < \psi_\lambda, \psi_\lambda > . \tag{39$'$}$$

So for any given function $\varphi(N)$ representation of such algebra is defined uniquely (up to unitary equivalence) if $Sp\, N$ is fixed.

We have proved that if conditions (17) -(19) are fulfilled, then SpN is district and consists of numbers $\lambda + k$, $k \in \mathbf{Z}$. Now let's show that if N has such a spectrum, then operator $U(s)$ with necessary properties really exists. For this it is sufficient to determine $U(s)$ on vectors $\psi_{\lambda+k}$ as K is a span of these vectors. If we introduce $U(s)\psi_{\lambda+k} = exp\,(is(\lambda+k))\psi_{\lambda+k}$, we can check that $U(s)$ has all necessary properties (for details see [27]).

Let's stress that in Hilbert space conditions (17) - (19) are equivalent to Rellich-Dixmier conditions of regularity.

In the case of arbitrary finite numbers of degrees of freedom we suppose the existence of operator $U_i(s)$ with properties (17) - (19) for any i. Thus previous consideration can be extended to the case of arbitrary n.

9. CCR in case of infinite numbers of degrees of freedom

The properties of CCR in the case of infinite numbers of degrees of freedom in many respects differ from CCR properties in the case of finite numbers of degrees of freedom. We describe this case briefly. The properties of Weyl relations are described explicitly in books [9, 28, 29] and we don't consider them. We point out only that in GQFT non regular representations of Weyl relations are important as well [30]. Let's mention that in general the time evolution does not leave the Weyl algebra stable [31]. Most of the standard treatment of QFT models deals with the canonical variables directly, therefore we consider the problem of analyzing the representations of the Heisenberg algebra directly. Let's concentrate our attention on the properties of operator N:

$$N = \sum_{i=1}^{\infty} N_i \qquad N_i = a_i^+ a_i. \tag{45}$$

We admit that for any i usual regularity properties are satisfied.
Thus any N_i has the same spectrum as in case of finite numbers of degrees of freedom. In fact it is sufficient to show that condition (25) is valid as before. Indeed it follows from (1) and (24) that

$$[U_i(2\pi), a_j] = 0; \qquad [U_i(2\pi), a_j^+] = 0 \quad \forall i, j. \tag{24'}$$

Condition (25) is the consequence of these equations and irreducibility of the representation under consideration (see Section 7).

Let's consider only a positive metric space case. It is evident that the existence of operators $N_i \equiv a_i^+ a_i$, doesn't automatically lead to the existence of N. The problem of existence of operator N first were investigated in paper of Friedrichs [32]. Then representations of CCR were studied by Gårding and Wightman [33]. They found that up to unitary equivalence there exists only one (Fock) representation, where operator N exists (this statement has been proved in [34]).

It should be noted that rigorous consideration of the properties of N includes subtle points as we deal with the infinite sequence of unbounded operators. These problems were studied in papers of Dell'Antonio, Doplicher and Ruelle [35] and Chaiken [36]. For example the existence of N depends on the notion of the convergence. We avoid this problems by construction space H_0, in which every vector satisfies the following condition: $N_i x \neq 0$ only for finite numbers of i. Thus sum in (45) is a finite sum for every $x \in H_0$ and operator N exists. Below we consider CCR in this space.

For any operator N_i we have: $SpN_i = \mathbb{N}$ and thus $SpN = \mathbb{N}$. Moreover, it follows directly from (1) and (20) that

$$Na_i = a_i(N-1). \tag{46}$$

So, if

$$N\psi = n\psi, \tag{47}$$

then

$$Na_i\psi = (n-1)a_i\psi. \tag{48}$$

Continuing this process we obtain ψ_0 such that

$$N\psi_0 = 0. \tag{49}$$

Eq.(49) implies that

$$a_i\psi_0 = 0. \qquad \forall i \tag{50}$$

Indeed, if $a_i\psi_0 \neq 0$, then in accordance with (48)

$$Na_i\psi_0 = -a_i\psi_0. \tag{51}$$

But in positive metric space N has no negative eigenvalues.
In fact, if $N\psi = -\beta\psi, \quad \beta > 0$, then on the one hand $< \psi, N\psi >= -\beta < \psi, \psi >$, on the other hand $< \psi, N\psi >= \sum\limits_{i=1}^{\infty} < a_i\psi, a_i\psi >$.

Thus the generalized condition (12) is fulfilled and there exists a cyclic vacuum vector. Let's point out that vectors $(a_i^+)^m (a_j^+)^n \ldots (a_k^+)^l \psi_0$ form the basis in the space H_0.

In order to prove that all representations in which a cyclic vacuum vector exists are unitary equivalent it is sufficient to notice that all scalar products in H_0 are fully determined by CCR. Thus the proof of the unitary equivalence of these representations is the direct extension of the proof made in Section 8 for the case $n = 1$.

Let's point out that H_0 is not a closed space. It is a pre-Hilbert space. By usual way we can construct closure of this space, but this question is out of the scope of this report.

In conclusion we turn our attention to the representations for which operator N doesn't exist. Roughly speaking it means that $n = \infty$ in eq.(47), so that every state contains infinite number of particles. Evidently here we consider CCR in space differing from H_0. These representations are called strange. (Strictly speaking, as it is pointed out in [36], strange representations can be also realized in spaces, where operator N exists, but all vectors correspond the states with infinite numbers of particles.) Strange representations can't be avoid in QFT owing to their connection with Haag's theorem (see e.g. [37, 38]). The examples of such representations have been described in [37, 36], see also the book of Segal [39].

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