SECRET SYMMETRY OF QCD VACUUM

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A review of a secret symmetry of the cold QCD realm is presented. The Regge trajectories of hadrons prove to be related to infinite unitary multiplets of the $\mathrm{SL}(4, R)$ group. This group was suggested by Ne'eman and Šijački for an exhaustive phenomenological classification of hadrons. We discuss exact retarded solutions to the classical $\mathrm{SU}(N)$ Yang-Mills equations with the source composed of several colored point particles. Relying on features of the solutions, it is shown that the color gauge group of the background field generated by two- and three-quark clusters is just $\mathrm{SL}(4, R)$. The simultaneous consideration of $\mathrm{SU}(N)$, $\mathrm{SO}(N)$, and $\mathrm{Sp}(N)$ as gauge groups offers a plausible explanation of the fact that clusters containing two or three quarks are more stable than multiquark clusters.

By now, there are reasons to believe that the strongly interacting matter exists in two phases which are referred to as the hadronic phase and the quark-gluon plasma phase (for a review, see Meyer-Ortmanns, 1996). The phase transition is expected to occur at temperatures about 200 MeV or/and densities of some units of the normal nuclear density.

The phases differ energetically, but the most outstanding distinction resides in their symmetry properties. It is a common knowledge that the chiral symmetry broken in the hadronic phase is regained above the critical point. However, the chirality criterion is unsound for distinguishing these phases since quarks have finite mass in both phases. Moreover, we deal actually with two different phase transitions; the deconfinement is associated with large quark masses while the chiral symmetry restoring transition is attributed to zero quark mass limit. It is not yet clear whether these transitions persist for finite quark masses or whether they will occur together (though lattice results indicate that both transitions occur at the same critical point). One should, therefore, suggest another dissimilarity in symmetry.

At high temperatures, the asymptotical freedom dominates, hence the symmetry inherent in the plasma phase must be nothing but the conventional color symmetry $\mathrm{SU}(3)_c$. One usually consider $\mathrm{SU}(3)_c$ to be unbroken in both phases. Should this be the case, what is the symmetry of the hadronic phase?

The most striking feature of the hadronic phase is the clusterisation of quarks. Every quark must be contained in some meson or barion. The hadronic world could be endowed with order by means of the Regge trajectories shown as straight lines with a fixed slope on the Chew-Frautschi plot of the mass squared M^2 versus the angular momentum J; hadrons belonging to some Regge trajectory are separated by intervals $\Delta J = 2$. Besides, clusters containing more than three quarks reveal themselves in nuclei (Baldin, 1977), being though less stable than hadrons. On the other hand, a quark-gluon plasma lump is assumed to be color-neutral and, to a good approximation, free of the cluster structure.

One usually think of the clusterisation as a dynamical effect. However, a more fundamental standpoint is to examine the clusterisation as a manifestation of some symmetry. Let us begin with the fact that Ne'eman and Šijački developed an exhaustive phenomenological classification of hadrons where every Regge sequence is associated with an infinite multiplet of the $\mathrm{SL}(4, R)$ group (Ne'eman and Šijački, 1988, 1993). This hints that $\mathrm{SL}(4, R)$ directly concerns with the invariance properties of the hadronic phase. Where did this $\mathrm{SL}(4, R)$ come from? Dothan, Gell-Mann and Ne'eman (Dothan *et al.*, 1965) were the first to describe the Regge trajectories of mesons by infinite-dimensional unitary representations of SL$(3, R)$. This group is generated by the angular momentum operators L_i and the quadrupole operators T_{ij} with the commutation relations

$$[L_i\,,\,L_j] = i\epsilon_{ijk}\,L_k, \tag{1}$$

$$[L_i\,,\,T_{jk}] = i\epsilon_{ijl}\,T_{lk} + i\epsilon_{ikl}\,T_{jl}, \tag{2}$$

$$[T_{ij}\,,\,T_{kl}] = -i\,(\delta_{ik}\epsilon_{jlm} + \delta_{il}\epsilon_{jkm} + \delta_{jl}\epsilon_{ikm})\,L_m. \tag{3}$$

The Lie algebra sl$(3, R)$ represents the minimal scheme capable to explain two features of Regge trajectories: The $\Delta J = 2$ rule and the apparently infinite sequence of hadronic states. (We mention in passing that the commutation relations of L_i and T_{ij} resemble those of the number operator n and the creation and annihilation operators a and a^+ in the harmonic oscillator problem,

$$[n\,,\,a] = -a, \qquad [n\,,\,a^+] = a^+.$$

Just as a and a^+ shift occupation numbers by one unite, so T_{ij} raises or lowers eigenvalues of the angular momentum squared by two units).

It was found (Dothan *et al.*, 1965) that two infinite unitary representations belonging to the ladder series

$$D^{\rm ladd}_{SL(3,R)}(0;R): \quad \{J\} = \{0, 2, 4, \ldots\}, \qquad D^{ladd}_{SL(3,R)}(1;R): \quad \{J\} = \{1, 3, 5, \ldots\}$$

are associated with the π and ρ trajectories. In addition, there exists a unique spinorial ladder representation related to the N trajectory

$$D^{\rm ladd}_{SL(3,R)}(\frac{1}{2};R): \quad \{J\} = \{\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots\}$$

while the representation starting with $J = \frac{3}{2}$ belongs to the discrete series (яgievetskii and Sokachev, 1975).

$$D^{\rm disc}_{SL(3,R)}(\frac{3}{2};R): \quad \{J\} = \{\frac{3}{2}, \frac{5}{2}, \frac{7^2}{2}, \frac{9^2}{2}, \frac{11^2}{2}\ldots\}.$$

Thus the SL$(3, R)$ scheme, being usefully applied to the Regge trajectories of mesons, turns out to be inadequate to account for those of baryons.

It was assumed (Ne'eman and Šijački, 1985) that matters can be improved by a simultaneous application of sl$(3, R)$ and so$(1, 3)$. The commutation relations can be closed by embedding two algebras in sl$(4, R)$, a "relativistic generalization" of sl$(3, R)$. With adopting this SL$(4, R)$, one can utilize SO(4), the maximal compact subgroup of SL$(4, R)$ as a basis with J^P content of some $(j_1,\, j_2)$ representation:

$$J^P = (j_1 + j_2)^P,\; (j_1 + j_2 - 1)^{-P}, \ldots, (|j_1 - j_2|)^{\pm P}.$$

The operator T_{ij} shifts SO(4) multiplets in (j_1, j_2) by $\Delta j_{1,2} = 2$, and the structure of Regge sequences is reproduced by such shifting. To put it differently, the SU$(3)_f$ octet states are classified according to the $D^{\rm disc}_{SL(4,R)}(\frac{1}{2}, 0) \oplus D^{\rm disc}_{SL(4,R)}(0, \frac{1}{2})$ representation while the symmetrized product of this reducible representation and the finite-dimensional SL$(4, R)$ representation $(\frac{1}{2}, \frac{1}{2})$ is used for the decuplet states. A remarkable fact is that we have arrived at all kinds of hadrons with different total angular momenta J, the half-integer including. Although this scheme is quite restrictive, it is in a good agreement with known data of hadronic spectroscopy (Ne'eman and Šijački, 1988, 1993).

Thus the symmetry of the hadronic phase proves to be SL$(4,R)$ while that of the plasma phase is SU$(3)_c$. These groups differ greatly in mathematical properties. SL$(4,R)$ is a non-compact group with fifteen essential parameters. SU(3) is a compact group with six essential parameters. SU(3) is not a subgroup of SL$(4,R)$. Moreover, physically, these symmetries are quite different. SU$(3)_c$ is a group of internal symmetry whereas SL$(4,R)$ – in Ne'eman and Šijački view – operates in spacetime. The situation is terrible: A spontaneous symmetry breakdown scenario is inconceivable, any link between these symmetries is seemingly out of the question. (To my best knowledge, Dothan and Gell-Mann never returned to this bizarre symmetry of the hadronic realm).

It remains, however, to attempt tailoring SL$(4,R)$ as the structure group to a fiber bundle construction of the so-called strong gravity*. There are two reasons for this. First, the mass quadrupole operator of a K-particle cluster is

$$Q_{ij}=\sum_{I=1}^{K} m_I\,(\mathbf{x}_i^I\,\mathbf{x}_j^I-\frac{1}{3}\,\mathbf{x}_I^2\,\delta_{ij})$$

and its time derivative is

$$\dot{Q}_{ij}=\sum_{I=1}^{K}[\mathbf{x}_i^I\,\mathbf{p}_j^I+\mathbf{x}_j^I\,\mathbf{p}_i^I-\frac{2}{3}\,(\mathbf{x}^I\cdot\mathbf{p}^I)\,\delta_{ij}],$$

where $\mathbf{p}_i^I=m_I\,\mathbf{v}_i^I$. Thus $T_{ij}\equiv\dot{Q}_{ij}$ are just the above discussed quantities since their commutators, with taking into account the canonical commutation relations of $\mathbf{x}^I$ and $\mathbf{p}^I$, give rise to the sl$(3,R)$ algebra, Eqs.(1)–(3), (Weaver and Biedenharn, 1970). This realization of the operator T_{ij} implies that the infinite unitary SL$(3,R)$ multiplets are collections of excited hadron states originating from appropriate ground states due to absorption of spin-2 quanta mediating strong interactions.

Second, such an origin of T_{ij} accounts for the possibility of classical radiation of a tensor field in view of the well-known Einstein formula for the rate of gravitational energy emission,

$$\frac{dE}{dt}=\frac{k}{45}\,(\dddot{Q}_{ij})^2.$$

The quadrupole excitations of hadrons cannot be produced by the conventional binding mechanism attributed to the exchange of mesons of spin 0 or 1. Before the rise of QCD, an *ad hoc* "strong gravity" hypothesis (Isham *et al.*, 1971) was tried in which the f^0 meson with $J=2^+$ and the mass 1270 Mev was given a central role as the "strong graviton". This idea was left once it became clear that the mesons of spin 2 are usual quark-antiquark systems which, being too massive, are not competitive with pions and other agents capable of producing a strong coupling at low energy range.

Ne'eman and Šijački suggested a tensor operator

$$G_{\mu\nu}=\mathrm{tr}\,(A_\mu\,A_\nu)$$

(with A_μ being the gluon field potential) for the role of an effective agent of strong interactions in the infrared region (Ne'eman and Šijački, 1990, 1992). $G_{\mu\nu}$ was intended to provide a color-neutral two-gluon exchange between hadrons. This di-gluon construction represents a Riemanian metric emulating gravity since it preserves the Lorentz group,

$$D_\sigma\,G_{\mu\nu}=0.$$

*The availability of the Lorentz group SO$(1,3)$, a subgroup of SL$(4,R)$, in the fiber endows a formal resemblance of this fibration to a gauge version of the standard gravitation.

It behaves like graviton; it will stay massless because of Lorentz invariance, conservation of the energy-momentum tensor and Einstein covariance relating to pseudo-diffeomorphisms, the local $\mathrm{SU}(3)_c$ transformations. The idea was to show that the quantum algebra of the operators $G_{\mu\nu}$ Lontains $\mathrm{sl}(4, R)$ as a subalgebra.

Unfortunately, this enterprise remained incomplete. Ne'eman and Šijački failed to establish the ground state invariant under $\mathrm{SL}(4, R)$. But it is just this issue which is crucial for the justification of the strong gravity as a whole.

Indeed, it has long been known that the invariance of the vacuum is the invariance of world (Coleman, 1966): Once the ground state of a quantum-mechanical system is invariant under some group of symmetry, both Hamiltonian and commutation relations of this system reveal such invariance. However, this remarkable theorem does not tell us what is the invariance of vacuum by itself.

This point became more clear only in the mid-70th when the attention of field theorists was attracted to soliton-like solutions of classical field equations, and methods of the semiclassical quantization about a nontrivial classical background were sufficiently elaborated (Rajaraman, 1982). The vacuum of a given field was shown to be related to some kind of the Bose condensate of this field, quasi-particle excitations about it revealing themselves as field quanta. One may also imagine vacuum as a state with no quantum excitation of the given field, yet filled with its classical background. This led us to recognize that the invariance of the vacuum is nothing but the invariance of the classical background. Thus the responsibility for the $\mathrm{SL}(4, R)$ symmetry rests with the background described by some solution of the QCD equations in the classical limit. It is the classical background generated by quarks in hadrons that provides the $\mathrm{SL}(4, R)$ relief for gluon excitations.

Now I would like to present some results of the quest of the $\mathrm{SL}(4, R)$ gluon vacuum. An important observation was that the classical limit of QCD is in line with the limit of infinite number of colors (Yaffe, 1982; Das, 1987). In order to find the classical background, one should substitute $\mathrm{SU}(3)_c$ by $\mathrm{SU}(N)_c$ and derive the QCD field equations in the large N limit. This is a challenging task which is still unsolved. However, one may reasonably assume that the classical $\mathrm{SU}(N)$ Yang-Mills theory with large N is intimately related to the classical limit of QCD.

It should be point out that, given a theory with the action S invariant under $\mathrm{SU}(N)$, this automatically entails the invariance of S under $\mathrm{SL}(N, C)$. If we have no prior knowledge of the symmetry, it can be identified by the structure constants f_{abc} which appear in S. The specific values of f_{abc} entering into the action imply that S is invariant under $\mathrm{SU}(N)$. However, for any simple complex Lie algebra, there exists a basis, referred to as the Cartan basis, such that the structure constants are found to be real, antisymmetric and identical to the structure constants of the real compact form of this Lie algebra (see, e. g., Barut and Rączka, 1977). The basis of $\mathrm{su}(N)$ is simultaneously the Cartan basis of its complexification $\mathrm{sl}(N, C)$. Thus the presence of the structure constants of $\mathrm{SU}(N)$ in S needs not be the evidence for that the symmetry of S is $\mathrm{SU}(N)$; allowing for the complex-valued field variables, we enlarge the symmetry up to $\mathrm{SL}(N, C)$.

There are two classes of exact retarded solutions to the classical Yang-Mills equations (for a review and references, see Kosyakov, 1998). Solutions of either class are real valued and invariant under $\mathrm{SU}(N)$. By contrast, solutions of other class are complex valued with respect to the Cartan basis of $\mathrm{su}(N)$, but it is possible to convert them to the real form; in doing so the solutions would be invariant under $\mathrm{SL}(N, R)$ or its subgroups. In particular, the background generated by any three-quark cluster proves to be invariant under $\mathrm{SL}(4, R)$, and that generated by any two-quark cluster is invariant under $\mathrm{SL}(3, R)$. Since $\mathrm{SL}(3, R)$ is a subgroup of $\mathrm{SL}(4, R)$, the background field of every hadron is specified by the gauge group $\mathrm{SL}(4, R)$. This symmetry is independent of N and is retained in the limit $N \to \infty$.

Let us consider these findings in more detail. We are interesting in retarded solutions to the classical SU(N) Yang-Mills equations

$$D^\mu F_{\mu\nu} = 4\pi j_\mu, \tag{4}$$

where the source

$$j_\mu(x) = \sum_{I=1}^{K} \int d\tau_I \, Q_I(\tau_I) \, v_\mu^I(\tau_I) \, \delta^4[x - z_I(\tau_I)]$$

is composed of K classical point particles, "quarks", moving along timelike world lines $z_I^\mu(\tau_I)$ parametrized by the proper times τ_I, $v_\mu^I \equiv dz_\mu^I/d\tau_I$ is the four-velocity of Ith particle. Each particle is assigned a color charge $Q_I = Q_I^a T_a$ transforming as the adjoint representation of SU(N).

We begin with a retarded solution to the SU(2) Yang-Mills equations (4) in the single-quark case (Kosyakov, 1991),

$$A_\mu^a = \mp \frac{i}{g} \, \sigma_3^a \, \frac{v_\mu}{\rho} + \kappa \, (\sigma_1^a \pm i\sigma_2^a) \, R_\mu. \tag{5}$$

Here, g is the Yang-Mills coupling constant, σ_n Pauli matrices, κ arbitrary nonzero integration constant. The lightlike vector $R_\mu \equiv x_\mu - z_\mu^{\rm ret}$ is drawn from the point of emission $z_\mu^{\rm ret}$ to the point of observation x_μ, and $\rho \equiv R \cdot v$ the invariant distance between these points.

Note that Pauli matrices span the Cartan basis of su(2). Setting a new color basis

$$\mathcal{T}_1 \equiv i\sigma_1, \quad \mathcal{T}_2 \equiv \sigma_2, \quad \mathcal{T}_3 \equiv i\sigma_3,$$

and considering the parameter κ to be imaginary, one rearranges the potential (5) to the form $A_\mu = \mathcal{A}_\mu^a \, \mathcal{T}_a$ with real valued $\mathcal{A}_\mu^a$. Elements of the new basis are traceless imaginary valued 2×2 matrices satisfying the commutation relations of the Lie algebra sl($2, R$).

For $\kappa = 0$, the Yang-Mills equations (4) linearize, and one gets an Abelian solution

$$A_\mu^a = e_3 \, \sigma_3^a \, \frac{v_\mu}{\rho}, \tag{6}$$

(1') with e_3 being an arbitrary real constant.

Thus we get the retarded solutions (5) and (6) describing the single-quark background of two different phases. The first phase is specified by the gauge group SL($2, R$) while the second by the gauge group SU(2).

The extension to SU(N) offers no significant changes in the single-quark potential: In addition to a non-Abelian term [of the type of Eq.(5)] built out of a triplet of color vectors which span the basis of the Lie algebra su(2), a subalgebra of su(N), there is an Abelian term decoupled from the remainder of the solution in that they commute. he adequacy of the initial gauge group SU(2) in the single-quark case is thus confirmed.

It is no great surprise that SL($2, C$) stands out against SL(N, C), $N > 2$, in the single-quark case. The metrical structure of the base embodied in the Lorentz group SL($2, C$) is all that should be mapped by the future light cone into the fiber, so that the color space SL($2, C$) is the only exact image. Proceeding from SO(N) or Sp(N) rather than SU(N), one reaches the same single-quark solution due to the isomorphisms

$$\text{su}(2) \sim \text{so}(3) \sim \text{sp}(1) \qquad \text{and} \qquad \text{sl}(2, R) \sim \text{su}(1, 1) \sim \text{so}(2, 1) \sim \text{sp}(1, R),$$

or, more generally,

$$\text{sl}(2, C) \sim \text{so}(2, C) \sim \text{sp}(1, C).$$

Next, turn to the two-quark case (Kosyakov, 1993, 1994). We adopt SU(3), the minimal group whereby the retarded field generated by two bound quarks is constructed. The point is that the field of a bound quark occupies individually some "elementary" $SL(2,R)$ cell of the color space while $SL(3,C)$, the complexification of SU(3), contains two such cells.

One usually realizes su(3) with the aid of the Gell-Mann matrices $T_a = \lambda_a/2$. However, it is more convenient for our purposes to use an overcomplete color basis spanned by the nonet of 3×3 matrices including three diagonal matrices

$$H_1 \equiv \frac{1}{2}\,(\lambda_3 + \frac{\lambda_8}{\sqrt{3}}) = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_2 \equiv -\frac{1}{2}\,(\lambda_3 - \frac{\lambda_8}{\sqrt{3}}) = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$H_3 \equiv -\frac{\lambda_8}{\sqrt{3}} = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{7}$$

(H) which are related by

$$\sum_{n=1}^{3} H_n = 0,$$

and six raising and lowering matrices

$$E_{12}^{+} \equiv \frac{1}{2}\,(\lambda_1 + i\lambda_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{12}^{-} \equiv E_{21}^{+} \equiv \frac{1}{2}\,(\lambda_1 - i\lambda_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{13}^{+} \equiv \frac{1}{2}\,(\lambda_4 + i\lambda_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13}^{-} \equiv E_{31}^{+} \equiv \frac{1}{2}\,(\lambda_4 - i\lambda_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$E_{23}^{+} \equiv \frac{1}{2}\,(\lambda_6 + i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{23}^{-} \equiv E_{32}^{+} \equiv \frac{1}{2}\,(\lambda_6 - i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{8}$$

(E) Given this color basis, three retarded solutions are

$$A_\mu^{(1)} = \mp\frac{2i}{g}\,(H_1\,\frac{v_\mu^1}{\rho_1} + H_2\,\frac{v_\mu^2}{\rho_2}) + \kappa\,(E_{13}^{\pm}\,R_\mu^1 + E_{23}^{\pm}\,R_\mu^2)\,\delta(R^1\cdot R^2). \tag{9}$$

(2.1)

$$A_\mu^{(2)} = \mp\frac{2i}{g}\,(H_3\,\frac{v_\mu^1}{\rho_1} + H_1\,\frac{v_\mu^2}{\rho_2}) + \kappa\,(E_{32}^{\pm}\,R_\mu^1 + E_{12}^{\pm}\,R_\mu^2)\,\delta(R^1\cdot R^2). \tag{10}$$

(2.2)

$$A_\mu^{(3)} = \mp\frac{2i}{g}\,(H_2\,\frac{v_\mu^1}{\rho_1} + H_3\,\frac{v_\mu^2}{\rho_2}) + \kappa\,(E_{21}^{\pm}\,R_\mu^1 + E_{31}^{\pm}\,R_\mu^2)\,\delta(R^1\cdot R^2). \tag{11}$$

(2.3) They represent actually the same Yang-Mills field being related by gauge transformations.

The solutions (9)–(11) become real-valued with respect to the color basis

$$\mathcal{T}_1 \equiv i\frac{\lambda_1}{2}, \quad \mathcal{T}_2 \equiv \frac{\lambda_2}{2}, \quad \mathcal{T}_3 \equiv i\frac{\lambda_3}{2}, \quad \mathcal{T}_4 \equiv i\frac{\lambda_4}{2},$$

$$\mathcal{T}_5 \equiv \frac{\lambda_5}{2}, \quad \mathcal{T}_6 \equiv i\frac{\lambda_6}{2}, \quad \mathcal{T}_7 \equiv \frac{\lambda_7}{2}, \quad \mathcal{T}_8 \equiv i\frac{\lambda_8}{2},$$

or else

$$\mathcal{H}_n \equiv i\, H_n, \qquad \mathcal{E}^{\pm}_{mn} \equiv i\, E^{\pm}_{mn}.$$

With reference to the explicit form of H_n and $E^{\pm}_{mn}$, Eqs.(7)–(8), one finds that $\mathcal{H}_n$ and $\mathcal{E}^{\pm}_{mn}$ as well as $\mathcal{T}_n$ are traceless imaginary 3×3 matrices satisfying the commutation relations of the Lie algebra sl$(3, R)$. hus the gauge symmetry of the solutions (9)–(11) is SL$(3, R)$.

For $\kappa = 0$, a retarded solution is a superposition of two single-quark potentials (6),

$$A_\mu = \sum_{I=1}^{2}\sum_{n=1}^{3} e^I_n\, H_n\, \frac{v^I_\mu}{\rho_I}. \tag{12}$$

The gauge group of this solution is SU(3).

Thus we have the solutions (9)–(11) and (12) describing the two-quark backgrounds of two different phases. The first phase is specified by the gauge group SL$(3, R)$ containing two "elementary" color cells SL$(2, R)$, while the second by the gauge group SU(3).

Starting from SO(N) or Sp(N) in the two-quark case, one arrives at other results as opposed to SU(N). Both the so$(4, C)$ and so$(5, C)$ color spaces are suitable for an accommodation of two "elementary" color cells so$(3, C) \sim$ sl$(2, C)$. But so$(4, C)$ is not semisimple, and the Cartan-Killing metric is singular here. As for so$(5, C)$, it is isomorphic to sp$(2, C)$, and we envisage two alternatives in the description of the color space in the two-quark case, either sl$(3, C)$ or so$(5, C) \sim$ sp$(2, C)$.

The discussion of solutions to Eq.(4) with the source composed of K quarks echoes in many respects that in the two-quark case (Kosyakov, 1998). Starting from the gauge group SU(N), one obtains two types of solutions corresponding to two phases of the strongly interacting matter. One can show that solutions with the noncompact gauge group SL$(K+1, R)$ correspond to the background generated by K bound quarks in the cold phase, while solutions invariant under the initial compact gauge group SU(N) describe the background generated by quarks forming a plasma lump in the hot phase.

The emergence of a classical solution invariant under a noncompact gauge group different from the initial one is a new field-theoretic phenomenon named "spontaneous symmetry deformation" (Kosyakov, 1994, 1998). This phenomenon contrasts with the famous spontaneous symmetry breakdown in three aspects.

First, we have no solution invariant under a subgroup of the initial gauge group; we deal instead with solutions invariant under two different real forms of the complexification of the initial group.

Second, both solutions are now stable against small disturbances despite the solution with a noncompact symmetry group is more advantageous energetically than that with the compact symmetry group. Third, the critical point $\kappa = 0$ is independent of parameters appearing in the action, as opposed to the Yang-Mills-Higgs theory where the spontaneous symmetry breakdown is directly related to parameters controlling the convexity of the Higgs potential.

A close look at the background generated by two-quark clusters, Eqs.(9)–(11), shows that, in the cold phase, the background of each quark individually occupies some "elementary" sl$(2, R)$ cell. Neither of two backgrounds generated by different quarks may be contained in the same sl$(2, R)$. This is similar to the Pauli blocking principle. Just as a cell of volume h^3 in the phase space might be occupied by at most one fermion with a definite spin polarization, so any sl$(2, R)$ cell is intended for a background of only one quark. Choosing SO(N) or Sp(N) rather than SU(N), one singles out the same color sell so$(2, 1) \sim$ sp$(1, R) \sim$ sl$(2, R)$.

On the other hand, in the hot phase, assuming the total color charge of quarks in a given plasma lump to be zero, the parameters e^I_n in the solution like that of Eq.(12) are to be appropriately fitted. Then the most advantageous field configuration is such that the color charges of quarks are lined up into a fixed color direction, thereby reducing SU(N) to SU(2). This bears resemblance to the Bose-Einstein condensation in the color space.

Thus the "color Pauli principle" preventing a body of $K+1$ color cells against shrinkages witnesses that the large-N limit is a self-consistency condition in the cold world containing many bound quarks. Meanwhile the "color Bose-Einstein condensation" suggests the sufficiency of SU(2) for the hot phase. Although one fails to reduce SU$(N)_c$ to SU$(3)_c$, this phenomenon shows promise of regaining SU$(3)_c$ in a more realistic model.

Let us touch on the three-quark case. There is no need for writting out the corresponding solutions explicitly. We only remark that, starting from SU(N), one finds the minimal simple complex group containing three "elementary" color cells to be SL$(4, C)$. Trying SO(N) as the gauge group, one arrives at the same result since so$(6, C) \sim$ sl$(4, C)$. Starting from Sp(N), the gauge symmetry of the background generated by a three-quark cluster turns out to be Sp$(3, C)$.

The fact that the gluon vacuum in the presence of two and three bound quarks has the gauge symmetry SL$(4, R)$ is in agreement with the Ne'eman and Šijački findings. But, in the present context, an additional degeneracy of Regge multiplets is the case. This is due to the existence of alternative gauge symmetries of the gluon vacuum in the two-quark and three-quark cases, SO$(3, 2) \sim$ Sp$(2, R)$ and Sp$(3, R)$, respectively.

Notice that SL$(4, R)$ of Ne'eman and Šijački operates in spacetime while the present SL$(4, R)$ acts in the color space. However, it is conceivable that two arenas interweave; gluon excitations about the background with the SL$(4, R)$ color symmetry will manifest themselves as if their color degrees of freedom were converted into spin degrees of freedom described by irreducible unitary representations of the Lorentz group SO$(1, 3)$, a subgroup of SL$(4, R)$. A conversion of isospin into spin in gauge theories discovered in the mid-70th (Jackiw and Rebbi, 1976; Hasenfratz and 't Hooft, 1976) seems to be of the direct relevance to our discussion. This phenomenon has its origin in a combination of some singular gauge field of magnetic type, such as the magnetic field generated by a monopole, and an isospin-degenerate field which is the source of a Coulomb-like electric field.

However, this similarity is not quite complete. An external color field with an appropriate SL$(4, R)$ degeneracy generating a long-range counterpart of the initial background field is hardly adoptable to the present constructions. On the other hand, restricting the consideration to a pure Yang-Mills system, one faces infrared divergences due to certain excitation modes.

We regard every K-quark cluster on the equal footing. It is well known, however, that hadrons are much more stable than multiquark clusters. One may wonder what a plausible explanation of this fact may be. We can envision consecutive constructions of the classical Yang-Mills systems, with the color spaces SL(N, C), SO(N, C), and Sp(N, C). There is nothing to decide between these alternatives; the "elementary" color cell is the same for any choosing. Hence all should persist and interfere. Is there the largest color cell outside of which three classical pictures become quite different? Such a sell does correspond to the three-quark case. It should be recognized that the interference of distinct color backgrounds is responsible for the splitting of energetical levels, which leads to the decay of clusters. No interference occurs in the single-quark case because sl$(2, C) \sim$ so$(3, C) \sim$ sp$(1, C)$. In the two-quark and three-quark cases, two alternatives interfere, respectively, sl$(3, R)$ and so$(3, 2) \sim$ sp$(2, R)$, and sp$(3, R)$ and sl$(4, R) \sim$ so$(3, 3)$. Thus clusters with two or three quarks are moderately stable. For $n > 4$, there are no isomorphisms between members of the series sl(n, C), so$(2n-1, C)$, sp(n, C), and so$(2n, C)$. Therefore, the interference of three alternatives keeps multiquark clusters away from stability.

These considerations based on the existence of only four classical Cartan series of simple complex continuous groups, A_n, B_n, C_n, and D_n, or, in more usual physical notations, $\mathrm{SL}(N, C)$, $\mathrm{SO}(2n+1, C)$, $\mathrm{Sp}(N, C)$, and $\mathrm{SO}(2n, C)$, are quite general. I do not know any other explanation of the fact that only two-quark and three-quark clusters might be moderately stable.

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References

[1] Baldin A.M. “The physics of relativistic nuclei.” Sov. J. Part. Nucl. 1977. **8**, 175.

[2] Barut A.O. and R. Rączka. *Theory of Group Representations and Applications.* 1977. (PWN – Polish Scientific Publishers, Warszawa).

[3] Coleman S. “The invariance of the vacuum is the invariance of world,” J. Math. Phys. 1966. **7**, 787.

[4] Das S. R. “Some aspects of large-N theories.” Rev. Mod. Phys. 1987. **59**, 235.

[5] Dothan Y., M. Gell-Mann and Y. Ne’eman. “Series of hadron energy levels as representations of non-compact group .” Phys. Lett. 1965. **17**, 148.

[6] Hasenfratz P. and G. ’t Hooft. “Fermion-Boson Puzzle in a Gauge Theory.” Phys. Rev. Lett. 1976. **36**, 1119.

[7] Isham C.J., A. Salam and J. Strathdee. “f-dominance of gravity.” Phys. Rev. 1971. D **3**, 867.

[8] Jackiw R. and C. Rebbi. “Spin from Isospin in a Gauge Theory.” Phys. Rev. Lett. 1976. **36**, 1116.

[9] Kosyakov B.P. “Field of arbitrarily moving colored charge.” Theor. Math. Phys. 1991. **87**, 632.

[10] Kosyakov B.P. “Classical Yang-Mills field generated by two point colored charges.” Phys. Lett. 1993. **B 312**, 471.

[11] Kosyakov B.P. “Exact solutions of the Yang-Mills equations with the source in the form of two point color charges,” Theor. Math. Phys. 1994. **99**, 409.

[12] Kosyakov B.P. “Exact solutions in the Yang-Mills-Wong theory.” Phys. Rev. 1998. **D 57**, 5032.

[13] Meyer-Ortmanns H. “Phase transitions in quantum chromodynamics.” Rev. Mod. Phys. 1996. **68**, 473.

[14] Ne’eman Y. and Dj. Šijački. “$\overline{\mathrm{SL}}(4, R)$ classification for hadrons.” Phys. Lett. 1985. **157 B**, 267.

[15] Ne’eman Y. and Dj. Šijački. “Hadrons in an $\overline{\mathrm{SL}}(4, R)$ classification.” Phys. Rev. 1988. **D 37**, 3267.

[16] Ne’eman Y. and Dj. Šijački. “QCD as an effective strong gravity.’ Phys. Lett. 1990. **247 B**, 571.

[17] Ne’eman Y. and Dj. Šijački. “Proof of pseudo-gravity as QCD approximation for the hadron IR region and $J \sim M^2$ Regge trajectories.” Phys. Lett. 1992. **276 B**, 173.

[18] Ne’eman Y. and Dj. Šijački. “Hadrons in an $\overline{\mathrm{SL}}(4, R)$ classification. II. Mesons and C, P assignments.” Phys. Rev. 1993. D**47**, 4133.

[19] яgievetskii, V.I. and E. Sokachev. “Primitive representationsof thesl(3,R) algebra.” Theor. Math. Phys. 1975. **23**, 845.

[20] Rajaraman R. *Solitons and Instantons.* (Noth-Holland, Amsterdam, 1982).

[21] Weaver L. and L.J. Biedenharn. “Nuclear rotational bands and $\mathrm{SL}(3, R)$ symmetry.” Phys. Lett. 1970. **32B**, 326.

[22] Yaffe L.G. “Large N limits as classical mechanics.” Rev. Mod. Phys. 1982. **54**, 407.