PHYSICS AND GEOMETRY

Yu.S. Vladimirov
Department of Physics, Moscow State University, Moscow, Russia

Multidimensional geometric models of physical interactions of Kaluza-Klein's type are considered. In these models, (1) for additional coordinates the topology of n-torus is used, (2) the additional coordinate dependence of the mixed components of the multidimensional metric is introduced and (3) the reduction to a 4-dimensional theory is carried out using the n-adic method in a gauge like the 4-dimensional chronometric one. It is shown that (a) it is possible to unify general relativity with the Weinberg-Salam theory of electroweak interactions in the framework of 7-dimensional geometric model, (b) the unification of general relativity and classical chromodynamics is possible in the framework of 8-dimensional geometric model.

1. Introduction

The gauge point of view on the nature of physical interactions is known to dominate in the 20th century physics. However, physics may have geometrical way.

The development of the geometric approach to the physical interactions was begun by A.Einstein, which suggected the geometric explanation of gravitation. In the 4-dimensional space-time of general relativity the square of the interval between two neighbouring points is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \qquad (\mu, \nu = 0, 1, 2, 3). \tag{1}$$

Einstein showed that the components of the metric tensor $g_{\mu\nu}$ replace Newton's single gravitational potential.

In 1919-1921, T.Kaluza [1] suggested a unified theory of gravitation and electromagnetism on the basis of a 5-dimensional curved space-time, in which the square of the 5-dimensional interval is

$$dI^2 = G_{AB} dx^A dx^B, \qquad (A, B = 0, 1, 2, 3, 5). \tag{2}$$

In this theory the mixed components of the 5-dimensional metric tensor $G_{5\mu}$ correspond to the electromagnetic vector potential A_μ.

The research into 5-dimensional theory developed unevenly. The interest in this problem rose and fell alternately. In the 1920s – 1930s, this theory was developed by H.Mandel, L. de Broglie, A.Einstein [2], P.Bergmann and other physicists.

Another branch of the 5-dimensional geometric models was developed by O.Klein [3], V.A. Fock [4], Yu.B.Rumer [5] and other physicists. Kaluza's theory and Klein's one deal with different additional dimensions. Nevertheless, all the multidimensional geometric models are admitted to be called Kaluza-Klein's.

There were a set of difficulties in the 5-dimensional models, which prevented from its development. The pioneer works of these trends proved to be premature. They revealed the problems unsolvable promptly, whereas the standard approach was less conjectural and allowed one to bypass acute problems. The situation changed in the 1970s when Weinberg-Salam's model of electroweak interactions was developed.

Physics could have advanced toward Kaluza-Klein's type multidimensional geometrical models, as demonstrated by a renewed interest inspired in them since late seventies and lasting for the last decades. The results of the gauge theories of physical interactions existing now may be achieved in the framework of these models.

In this article based on the our previous works [6, 7, 8, 9], it is shown that the geometric analogy of Weinberg-Salam's electroweak interaction model can be developed in the framework of the 7-dimensional curved space-time. This theory unified Einstein's theory of gravitation and the known electroweak interaction model.

Einstein's general relativity can be combined with the classical chromodynamics in the framework of the 8-dimensional geometric model [9, 18] of Kaluza-Klein's type.

2. Five-dimensional Kaluza's theory

Kaluza postulated that at the foundations of every mathematical description of the universe there is a curved five-dimensional world with one time and four spatial coordinates. In such a manifold, the metric tensor G_{AB} (each capital Latin index A, B, C, etc., can take the values 0, 1, 2, 3, 5) has 15 components. They correspond to the ten components of the 4-dimensional metric tensor $g_{\mu\nu}$, the four components of the electromagnetic vector potential A_μ, and one more as yet unindentified component G_{55}:

$$G_{AB} = \left(\begin{array}{cccc|c} G_{00} & G_{01} & G_{02} & G_{03} & G_{05} \\ G_{10} & G_{11} & G_{12} & G_{13} & G_{15} \\ G_{20} & G_{21} & G_{22} & G_{23} & G_{25} \\ G_{30} & G_{31} & G_{32} & G_{33} & G_{35} \\ \hline G_{50} & G_{51} & G_{52} & G_{53} & G_{55} \end{array}\right) \equiv \left(\begin{array}{c|c} G_{\mu\nu} & G_{\mu 5} \\ \hline G_{5\mu} & G_{55} \end{array}\right) \to \left(\begin{array}{c|c} g_{\mu\nu} & A_\mu \\ \hline A_\nu & G_{55} \end{array}\right). \tag{3}$$

The fifth dimension is different from the four classical ones. The space must be closed (compacted) in the fifth coordinate. The wave functions of a charge particles depend on the fifth coordinate as

$$\Psi = \psi(x^\mu)\exp(i\alpha\varepsilon_5 x^5) \equiv \psi(x^\mu)\exp\left(\frac{iec\varepsilon_5}{2\sqrt{k_g}\hbar}x^5\right), \tag{4}$$

where $\psi(x^\mu)$ is the part of the quantity, which depends only on the classical coordinates, e is the electrical charge of the electron, $\hbar$ is Planck's constant, k_g is the Newtonian gravitational constant, ε_5 is a dimensionless parameter. The period

$$T = \frac{2\pi}{\alpha} = 4\pi\sqrt{\frac{\hbar k_g}{c^3}} \cdot \sqrt{\frac{\hbar c}{e^2}} \simeq 10^{-31} cm \tag{5}$$

of the dependence of Ψ on x^5 is extremely short in comparison with the distance for which the standard equations hold.

In Kaluza theory it is supposed that the components of 5-dimensional metric tensor are indepent of the fifth coordinate:

$$\frac{\partial G_{AB}}{\partial x^5} = 0. \tag{6}$$

To express the 5-dimensional geometric quantities and expressions in terms of the conventional 4-dimensional concepts, it is necessary to use a reduction of the theory to the 4-dimensional space-time, i.e. to apply the 4+1 splitting procedure [2]. It is performed by means of the monad method in special gauge like the chronometric gauge [10, 11] used in general relativity. The monad method can be represented by 4 constituents: (a) an algebra of the monad method; (b) specification of monad physico-geometric tensors; (c) definition of monad derivative operators; (d) presentation of basic relations in the monad form, i.e., in terms of only 4-dimensionally projected tensor quantities, monad physico-geometric tensors and monad derivative operators.

(a) Algebra of the monad method. In a generally covariant form the 5-metric G_{AB} is presented as follows:

$$G_{AB} = g_{AB} - \lambda_A \lambda_B, \tag{7}$$

where λ_A is the 5-dimensional vector (monad). It is orthogonal to the 4-dimensional metric tensor g_{AB}.

In the gauge like the chronometric one, the vector λ^A is directed along the x^5 lines, i.e.

$$\lambda^A = \frac{G_5^A}{\sqrt{-G_{55}}} \to \lambda_B = \frac{G_{5B}}{\sqrt{-G_{55}}}; \quad g_{\mu\nu} = G_{\mu\nu} - \frac{G_{5\mu} G_{5\nu}}{G_{55}}. \tag{8}$$

In this gauge, the following class of coordinate transformations is selected from the whole set of admissible ones:

$$x'^5 = x'^5(x^0, x^1, x^2, x^3, x^5); \tag{9}$$

$$x'^\mu = x'^\mu(x^0, x^1, x^2, x^3). \tag{10}$$

The physically interpreted quantities are those which are invariant under the transformations (9) and covariant under the 4-dimensional transformations (10). These neighbours are satisfied by the 4-dimensional metric tensor and 4-dimensionally projected quantities, as well as scalars obtained by projection of tensor quantities onto the monad direction.

In such theory the electromagnetic vector potential is presented as follows:

$$A_\mu = \frac{c^2}{2\sqrt{k_g}} \lambda^5 \lambda_\mu \equiv -\frac{c^2}{2\sqrt{k_g}} \frac{G_{5\mu}}{G_{55}} \to G_{5\mu} = \frac{2\sqrt{k_g}}{c^2} A_\mu, \tag{11}$$

where we put $G_{55} = -1$; k_g is Newtonian gravitational constant.

(b) In the most general case, there are three **monad physico-geometric tensors.** However, taking into account (6) and $G_{55} = -1$, we have only one nonzero tensor

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \left(\frac{\partial \lambda_\nu}{\partial x^\mu} - \frac{\partial \lambda_\mu}{\partial x^\nu} \right) = \frac{\sqrt{k_g}}{c^2} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right), \tag{12}$$

which corresponds to the electromagnetic tensor strength.

(c) In the Kaluza theory a basic role is played by two **monad differential operators**:

$$\partial_5^\dagger = \lambda^A \frac{\partial}{\partial x^A} = \lambda^5 \frac{\partial}{\partial x^5}; \tag{13}$$

$$\partial_\mu^\dagger = g_\mu^A \frac{\partial}{\partial x^A} = \frac{\partial}{\partial x^\mu} + G_{5\mu} \frac{\partial}{\partial x^5}. \tag{14}$$

The operator (14) must be put into correspondence to the extended derivative in the standard theory of the electromagnetic interactions:

$$\partial_\mu^\dagger \Psi \to \left(\frac{\partial}{\partial x^\mu} + \frac{ieQ}{c\hbar} A_\mu \right) \psi. \tag{15}$$

Taking into account (4), we must postulate that the electrical charge of a particle is characterized by eigenvalue of the operator (13), i.e., the harmonics ε_5 in the exponential dependence of quantities on the fifth coordinate determines the value of the electric charge (in the units e):

$$Q = \varepsilon_5. \tag{16}$$

(d) Presentation of basic relations in the monad form leads to important results. We now enumerate the indoubtable successes ("Kaluza's miracles") of the 5-dimensional unified theory:

(1) the fifteen 5-dimensional "Einstein equations" in vacuum decompose into the standard system of the ten 4-dimensional equations of Einstein (of electrovacuum type), and system of the four standard Maxwell equations without sources on the right-hand sides, and, in general, one extra scalar equation;

(2) if the fifth coordinate is space-like, then we get a standard tensor for the energy-momentum of the electromagnetic field (with an appropriate sign) on the right-hand side of the 4-dimensional Einstein equations;

(3) four of the five geodesic equations are standard equations of motion for an electrically charged particle in gravitational and an electromagnetic field;

(4) the gradient (gauge) transformations in standard electrodynamics $A'_\mu \to A_\mu + \partial f/\partial x^\mu$ correspond to admissible transformations of the fifth coordinate, $x'^5 = x^5 + f(x^0, x^1, x^2, x^3)$.

This variant of the 5-dimensional theory did not gain the trust of physicists for some fairly reasons. They were enumerated in [6, 11].

3. Kaluza-Klein's theory

3.1. 5-Dimensional Klein-Fock-Rumer's theory

In another branch of 5-dimensional theory developed by O.Klein [3], V.A.Fock [4] and Yu.B.Rumer [5], it is postulated another dependence of quantities on the additional coordinate

$$\Psi = \psi(x^\mu)\exp(i\beta x^4) \equiv \psi(x^\mu)\exp\left(\frac{imc}{\hbar}x^4\right), \tag{17}$$

where m is the mass of a particle, β is a small parameter, characterizing the compactification period of the additional coordinate, noted as x^4. In the branch of 5-dimensional theory, it is also used the monad method. The physical meaning of the additional coordinate x^4 is the classical action.

As Rumer emphasized, the geometrization of the electromagnetic field was not a main intention of this theory. With the help of the fifth coordinate the mass terms were introduced in equations. However, it was temptation to introduce also the electromagnetic field with the help of the same additional coordinate.

The failure of Rumer's 5-dimensional theory was mainly due to the assumption that the 5-dimensional interval along the particle's path was assumed to vanish. This meant that the fifth component of five-velocity $d\xi/ds \equiv \xi_M dx^M/ds$ was unity, and hence the electrical charge of the particles could not be brought into the equations of motion. Rumer attemped to find a way out of this situation by identifying extra components of the five-metric with electromagnetic quantities, i.e.

$$\tilde{G}_{\mu 4} = \frac{q}{mc^2}A_\mu, \tag{18}$$

where e is the particle's electrical charge. However, space-time then became dependent upon the properties of a concrete particle, that is, it became configurational (each particle had its own space). Moreover, a universal space-time had also to be postulated beside the configurational one. How to combine the two spaces was a question which defeated him (and it may even be unanswerable in the framework of a 5-dimensional theory).

3.2. 6-Dimensional Kaluza-Klein's theory

If, however, Rumer's work is generalized to a 6-dimensional theory with two additional coordinates x^4 and x^5, the above-mentioned difficulties do not occur. In this theory quantities should have the following dependence on the additional coordinates:

$$\Psi = \psi(x^\mu)\exp(i\beta x^4 + i\alpha x^5), \tag{19}$$

where α and β corespond to ones in Eqs. (4) and (17).

In this model it is necessary to use a reduction of the theory to the 4-dimensional space-time, i.e. to apply the 4+1+1 splitting procedure to the original 6-dimensional manifold. It is performed by means of the dyad method in a special gauge like the twice chronometric gauge used in general relativity. The metric tensor G_{MN} of the 6-dimensional manofold may be written as

$$G_{MN} = g_{MN} - \xi_M\xi_N - \lambda_M\lambda_N, \tag{20}$$

where g_{MN} is the metric tensor of the classical space- time, ξ_M and λ_N are two 6-dimensional space-like vectors (dyad).

The interaction of a particle is described by the dyadic differential operator

$$\partial^{\dagger\dagger}_\mu = \frac{\partial}{\partial x^\mu} + (\xi^4\xi_\mu + \lambda^4\lambda_\mu)\frac{\partial}{\partial x^4} + \lambda^5\lambda_\mu\frac{\partial}{\partial x^5}, \tag{21}$$

which is invariant under the permissible transformations of additional coordinates and covariant under the 4-dimensional transformations (10).

The combination $\lambda^5\lambda_\mu$ is identified, as (11), with the electromagnetic vector potential A_μ. However, in this theory there is another combination $\xi^4\xi_\mu + \lambda^4\lambda_\mu$, which depends on the components $G_{4\mu}$. What is its physical meaning? It is difficult to answer this question in the frame of the geometrical paradigm. However, there are reasons [9] to assume that the combination is also proportional to the electromagnetic potential A_μ. Then the corresponding term in (21) means an additional small interactions of electromagnetic type.

4. 7-Dimensional geometric model of gravi-electroweak interactions

4.1. Main ideas and methods of the 7-dimensional model

To construct a multidimensional geometric model of gravi-electroweak interactions, it is necessary to solve the following problems:

1. Give an adequate geometric foundation of two quantized charges, namely, the hypercharge Y and the isotopic spin projection T_3.
2. Indicate a geometric image of four vector fields: B_μ and the triplet $A(s)_\mu$ (where s=1, 2, 3), the carriers of the electroweak interactions in the Weinberg-Salam model.
3. Show that the nonlinear expressions for the tensors describing the vector field strengths, corresponding to a non-Abelian nature of the $SU(2)$ group, arise naturally in the geometric model.
4. Show that the multidimensional scalar curvature makes it possible to obtain all the components of the Lagrangian density of four vector fields, possessing the $U(1)\times SU(2)$ symmetry, known from the Weinberg-Salam model.
5. Describe the fermion field doublet — the neitrino and the electron (for a single generation of leptons).

6. Show that the standard methods of describing spinors in curved space-time lead to the known expressions in the Lagrangian density of interaction of the fermion doublet with the intermediate vector bosons.

7. Indicate a geometric analogue of the known Higgs mechanism for the introduction of particle rest masses.

To solve all these and other problems, it is suggested to use the following geometric ideas and methods:

(i) It has proved to be necessary to increase the space-time dimension by two (plus one) units. Specifically, this is dictated by solving the first of the above problems. In multidimensial theory the charges are known to have up to constant factors, the meaning of momenta along the additional coordinates.

(ii) The extra dimensions should be compactified, i.e., closed with a very small period. It is suggested to use the simplest topology of a 3-torus. This requires a cyclic dependence of all quantities on the additional coordinates:

$$\Psi = \psi(x^\mu)\exp[i\beta x^4 + i\alpha(\varepsilon_5 x^5 + \varepsilon_6 x^6)], \tag{22}$$

where $\psi(x^\mu)$ is the part of quantities, both geometric and those introduced to the geometry from outside, which depend only on the classical coordinates, β and α are small parameters of dimension $[cm^{-1}]$, characterizing the extra-dimension compactification periods, ε_5 and ε_6 are dimensionless parameters. This distinguishes the present approach from the majority of others [12, 13] where the topology of a sphere was used (see also [14]).

(iii) In the 7-dimensional space with the considered signature $(+---|---)$ spinors must have 8 complex components [15]. In the theory under consideration this 8-component spinor Ψ is presented in terms of the conventional 4-component spinors describing an electron and a neitrino.

(iv) In accordance with the spirit of general relativity, the key (basic) expression is chosen as a 7-dimensional Lagrangian hyperdensity, consisting of a geometric part and the contribution of spinor matter:

$$\mathcal{L}_{(7)} = \sqrt{G^{(7)}}\left[\frac{{}^7R}{2\tilde{æ}c} + \frac{i\hbar c}{2}\overline{\Psi}\Gamma^M\nabla_M\Psi + (h.c.)\right], \tag{23}$$

where h.c. means the Hermitian conjugate expression; $G^{(7)}$ is the determinant of the matrix G_{MN} of the 7-dimensional metric tensor; ${}^7R$ is the 7-dimensional scalar curvature and the covariant derivative is

$$\nabla_M\Psi = \frac{\partial\Psi}{\partial x^M} - \frac{1}{4}\Delta_M(NP)\Gamma(N)\Gamma(P)\Psi, \tag{24}$$

where $\Delta_M(NP)$ are an 7-dimensional Ricci rotation coefficients. Γ_M are 8×8 matrices which satisfy

$$\Gamma_M\Gamma_N + \Gamma_N\Gamma_M = 2G_{MN}I_8 \tag{25}$$

with the index $M = 0, 1, 2, 3, 4, 5, 6$.

(v) An essentially new feature of this theory as compared with general relativity and the 5-dimensinal Kaluza theory is that it is allowed for some components of the metric to be complex. An explanantion is that in this theory it is suggested to describe the charged vector $W^\pm$ bosons (and the charged scalar Higgs bosons) in terms of metric components.

(vi) To introduce the particle rest masses, it is suggested to use a procedure similar to the Higgs mechanism. Herewith, the scalar fields are assumed to be stipulated by a conformal factor. This means a transition from the original 7-dimensional metric to a conformally corresponding one, where the conformal factor is expressed in terms of one of the additional diagonal components of the metric.

(vii) To express the multidimensional geometric quantities and expressions in terms of the conventional 4-dimensional concepts of the standard Weinberg-Salam model, it is necessary to use a reduction of the theory to the 4-dimensional space-time, i.e. to apply the 4+1+1+1 splitting procedure to the original 7-dimensional manifold. It is performed by means of the triad method in a special gauge like the thrice chronometric gauge used in general relativity.

Recall that the triad method, as well as the monad one, can be represented by 4 constituents.

$(viii)$ The final expression for the 4-dimensional Lagrangian density of gravi-electroweak interactions is obtained from the 7-dimensional hyperdensity (23) as a result of averaging over the small periods of extra coordinate dependence. After integration of the Lagrangian hyperdensity in dx^4, dx^5 and dx^6, all exponential terms like (22), which have not cancelled due to multiplication of components, disappers, thus resulting in an expression that depends only on the 4 classical coordinates.

4.2. The metric and the physico-geometric tensors

Using the ideas of the monad and dyad methods, let us expound the necessary information on the triad method of 4+1+1+1 splitting. We will follow the ordinary scheme: algebra, notation of physico-geometric tensors and the basic expressions in the tetrad form. The 7-dimensional metric tensor in the above chosen signature has the following form in the triad method:

$$G_{MN} = g_{MN} - \xi_M \xi_N - \lambda_M \lambda_N - \sigma_M \sigma_N, \tag{26}$$

where the triadic vectors ξ_M, λ_M and σ_M satisfy ortonormality conditions.

In a direct generalization of the 5-dimensional Kaluza theory, each new dimension will introduce a new real vector field. In the 7-dimensional approach there are three such vectors, namely, the 4-dimensional parts of three triadic 7-vectors, ξ_μ, λ_μ, σ_μ. However, in the Weinberg-Salam model there are four vector fields: two neutral ones (B_μ and $A(3)_\mu$) and two charges ones ($A(1)_\mu$ and $A(2)_\mu$). Hence, in a direct generalization of Kaluza theory it is possible obtain only neutral vector fields, and in order to introduce charged vector fields, it is necessary to go beyond the frames of traditional Kaluza-Klein theory — to admit that the triadic vectors contain terms depending in a cyclic manner on the additionsl coordinates. The coefficients by the corresponding harmonics should be identified with new (charged) vector fields. As was shown in [8, 9], we must put

$$\xi_\alpha = C_{40} B_\alpha + C_{43} A(3)_\alpha + C_4^+ W_\alpha^+ \exp[2i\alpha x^6] + C_4^- W_\alpha^- \exp[-2i\alpha x^6]; \tag{27}$$

$$\lambda_\alpha = C_{50} B_\alpha + C_{53} A(3)_\alpha + C_5^+ W_\alpha^+ \exp[2i\alpha x^6] + C_5^- W_\alpha^- \exp[-2i\alpha x^6]; \tag{28}$$

$$\sigma_\alpha = C_{60} B_\alpha + C_{63} A(3)_\alpha + C_6^+ W_\alpha^+ \exp[2i\alpha x^6] + C_6^- W_\alpha^- \exp[-2i\alpha x^6], \tag{29}$$

where C_{40}, $\cdots$ are some constants, W_α^+ and W_α^-, are connected with $A(1)_\alpha$ and $A(2)_\alpha$ by the standard relations $W_\alpha^\pm = (1/\sqrt{2})(A(1)_\alpha \mp A(2)_\alpha)$.

We can construct a number of 4-dimensional tensors from the components of the 7-dimensional metric tensor and their first-order derivatives. In the present case of greatest interest are three second-rank antisymmetric tensors:

$$F^{(4)}_{\alpha\beta} = \frac{1}{2} g_\alpha^M g_\beta^N (\xi_{M,N} - \xi_{N,M}); \quad F^{(5)}_{\alpha\beta} = \frac{1}{2} g_\alpha^M g_\beta^N (\lambda_{M,N} - \lambda_{N,M});$$

$$F^{(6)}_{\alpha\beta} = \frac{1}{2} g_\alpha^M g_\beta^N (\sigma_{M,N} - \sigma_{N,M}), \tag{30}$$

which correspond (but not directly) to two antisymmetric tensors of the Weinberg-Salam model:

$$F(B)_{\mu\nu} = \frac{\partial B_\nu}{\partial x^\mu} - \frac{\partial B_\mu}{\partial x^\nu}; \quad \vec{F}_{\mu\nu} = \frac{\partial \vec{A}_\nu}{\partial x^\mu} - \frac{\partial \vec{A}_\mu}{\partial x^\nu} - \frac{ig_2}{\hbar c}(\vec{A}_\mu \vec{A}_\nu - \vec{A}_\nu \vec{A}_\mu), \tag{31}$$

where the arrows designate the matrix nature of the corresponding quantities. Thus $\vec{A}_\mu$ contains the triplet of vector fields $A(s)_\mu$ according to the formula $\vec{A}_\mu = (1/2) \times \sum_{s=1}^{3} A(s)_\mu \sigma(s)$, where $\sigma(s)$ are the three Pauli matrices.

Special attention should be paid to the nonlinear components in the right-hand side of (31), connected with the non-Abelian nature of the gauge fields $A(s)_\mu$ in the Weinberg-Salam model. In the 7-dimensional theory the expressions (30) contain nonlinear terms of a similar type. To assure this, it is sufficient to note that the components g_α^M have the form

$$g_\alpha^\beta = G_\alpha^\beta = \delta_\alpha^\beta; \quad g_\alpha^6 = \sigma^6 \sigma_\alpha + \lambda^6 \lambda_\alpha + \xi^6 \xi_\alpha. \tag{32}$$

Writing out the tensors (30), taking into account the extra coordinate dependence of the triadic vectors and the coordinate transformation freedom (we can put $\lambda^6 \sim \sigma^4 \sim \sigma^5 \sim 0$), we have:

$$F_{\alpha\beta}^{(4)} = \frac{1}{2}\left[(\xi_{\alpha,\beta} - \xi_{\beta,\alpha}) + \sigma^6(\sigma_\beta \xi_{\alpha,6} - \sigma_\alpha \xi_{\beta,6})\right]; \tag{33}$$

$$F_{\alpha\beta}^{(5)} = \frac{1}{2}\left[(\lambda_{\alpha,\beta} - \lambda_{\beta,\alpha}) + \sigma^6(\sigma_\beta \lambda_{\alpha,6} - \sigma_\alpha \lambda_{\beta,6})\right]; \tag{34}$$

$$F_{\alpha\beta}^{(6)} = \frac{1}{2}\left[(\sigma_{\alpha,\beta} - \sigma_{\beta,\alpha}) + \sigma^6(\sigma_\beta \sigma_{\alpha,6} - \sigma_\alpha \sigma_{\beta,6})\right]. \tag{35}$$

In the triadic form the 7-dimensional scalar curvature is presented in the following way:

$${}^7R = {}^4R + F_{\alpha\beta}^{(4)} F^{(4)\alpha\beta} + F_{\alpha\beta}^{(5)} F^{(5)\alpha\beta} + F_{\alpha\beta}^{(6)} F^{(6)\alpha\beta} + \cdots. \tag{36}$$

Just the second, third and fourth terms on the right describe the contributions of the intermediate vector bosons to the Lagrangian hyperdensity.

Recall that the gauge vector fields in the Weinberg-Salam model give the following contribution to the Lagrangian density [16]:

$$\mathcal{L}_{v.f} = -\frac{1}{16\pi} F(B)_{\mu\nu} F(B)^{\mu\nu} - \frac{1}{8\pi} Sp \vec{F}_{\mu\nu} \vec{F}^{\mu\nu}, \tag{37}$$

where the last term is constructed according to (31).

The geometric contributions from the vector fields in the Lagrangian hyperdensity and the contributions of vector fields (37) of the Weinberg-Salam model chould coincide, i.e.:

$$-\frac{1}{4æ} \int \left[\left(F_{\alpha\beta}^{(4)} F^{(4)\alpha\beta} + F_{\alpha\beta}^{(5)} F^{(5)\alpha\beta} + F_{\alpha\beta}^{(6)} F^{(6)\alpha\beta}\right)\right] dx^6 = \mathcal{L}_{v.f}. \tag{38}$$

Expressing the geometric strength tensors in terms of triadic vectors according to the (30) and averaging over the compactification period of extra dimensions, we obtain the 10 conditions upon the coefficients of neutral and charged vector fields in (27) – (29).

There are two possibilities:

$$1) \quad \frac{4k_g}{c^4} - \left(\frac{g_2}{2\alpha\sigma^6(\hbar c)}\right)^2 > 0; \qquad 2) \quad \frac{4k_g}{c^4} = \left(\frac{g_2}{2\alpha\sigma^6(\hbar c)}\right)^2. \tag{39}$$

Let us choose the second case. From the conditions we obtain the following expressions for for the coefficients of neutral vector fields:

$$C_{60} = C_{43} = C_{53} = 0; \quad C_{63} = \pm\frac{2\sqrt{k_g}}{c^2}; \quad C_{40}^2 + C_{50}^2 = \frac{4k_g}{c^4} \tag{40}$$

and the following conditions for the coefficients of the charged vector fields:

$$C_6^+ C_6^- = -\frac{2k_g}{c^4}; \quad (C_4^+ C_6^- + C_4^- C_6^+)^2 + (C_5^+ C_6^- + C_5^- C_6^+)^2 = 0; \quad C_4^+ C_4^- + C_5^+ C_5^- = \frac{6k_g}{c^4}. \tag{41}$$

4.3. Triadic differential operators and neutral vector fields

Let us consider another component of the triad method: specification of differential operators in the gauge used. There are four such independent operators invariant under the permissible transformations of the additional coordinates and covariant under the 4-dimensional transformations:

$$\partial_4^{\dagger\dagger} = \xi^N \frac{\partial}{\partial x^N} = \xi^4 \frac{\partial}{\partial x^4}; \tag{42}$$

$$\partial_5^{\dagger\dagger} = \lambda^N \frac{\partial}{\partial x^N} = \lambda^5 \frac{\partial}{\partial x^5}; \tag{43}$$

$$\partial_6^{\dagger\dagger} = \sigma^N \frac{\partial}{\partial x^N} = \sigma^6 \frac{\partial}{\partial x^6}; \tag{44}$$

$$\partial_\alpha^{\dagger\dagger} = g_\alpha^N \frac{\partial}{\partial x^N} = \frac{\partial}{\partial x^\alpha} + \xi_\alpha \partial_4^{\dagger\dagger} + \lambda_\alpha \partial_5^{\dagger\dagger} + \sigma_\alpha \partial_6^{\dagger\dagger}. \tag{45}$$

The operator (45) can be put into correspondence to the extended derivative in the Weinberg-Salam model

$$\partial_\alpha^{\dagger\dagger}\Psi \to \left(I_2 \frac{\partial}{\partial x^\alpha} - \frac{ig_1}{\hbar c} I_2 \frac{Y}{2} B_\alpha - \frac{ig_2}{\hbar c} T(s) A(s)_\alpha \right) \psi, \tag{46}$$

where Ψ is an arbitrary field function. Hence it follows that the additional coordinate dependence corresponds to the existence of a particle's isotopic spin (T_3) or hypercharge Y.

We postulate that the interaction with the neutral field B_α is characterized by the operator (43). Hence the harmonic ε_5 in exponential dependence of the quantities on the additional coordinates determines the value of the hypercharge Y, i.e.

$$\varepsilon_5 = Y. \tag{47}$$

We postulate that the interaction with the triplet of vector fields $A(s)_\alpha$ is characterized by the operator (44). Consequently the factor ε_6 inthe exponents is equal to twice the isotopic spin value T_3 in the Weinberg-Salam model:

$$\varepsilon_6 = 2T_3. \tag{48}$$

Combining (43) and (44), we arrive at the universal formula for the electrical charge value (in the units e) in the present 7-dimensional model

$$Q = \frac{1}{2}Y + T_3 = \frac{1}{2}(\varepsilon_5 + \varepsilon_6), \tag{49}$$

corresponding to the well-known formula in the Weinberg-Salam model.

In the 7-dimensional model one can obtain geometric expressions for the electromagnetic vector potential, the Z-boson, the electric charge and the Weinberg angle. To do so, we shall indicate the extra coordinate dependence of the neutral Higgs boson (of geometric origin) in accordance to (43) and (44). It is easy to show that mass vector field (Z-boson) and massles vector field (electromagnetic field) are presented in the forms:

$$Z_\mu = \frac{c^2}{2\sqrt{k_g}} \frac{\lambda^5 \tilde{\lambda}_\mu - \sigma^6 \tilde{\sigma}_\mu}{\sqrt{(\sigma^6)^2 + (\lambda^5)^2}} = \frac{c^2}{2\sqrt{k_g}} (\tilde{\lambda}_\mu \sin\theta_W - \tilde{\sigma}_\mu \cos\theta_W); \tag{50}$$

$$A_\mu = -\frac{c^2}{2\sqrt{k_g}} \frac{\sigma^6 \tilde{\lambda}_\mu + \lambda^5 \tilde{\sigma}_\mu}{\sqrt{(\sigma^6)^2 + (\lambda^5)^2}} = -\frac{c^2}{2\sqrt{k_g}} (\tilde{\lambda}_\mu \cos\theta_W + \tilde{\sigma}_\mu \sin\theta_W), \tag{51}$$

where we have introduced the angle θ_W which corresponds to the Weinberg angle:

$$\sin\theta_W = \frac{\lambda^5}{\sqrt{(\sigma^6)^2 + (\lambda^5)^2}}; \quad \cos\theta_W = \frac{\sigma^6}{\sqrt{(\sigma^6)^2 + (\lambda^5)^2}}. \tag{52}$$

In (50) and (51) the tilde marks only the neutral part of the triadic vectors.

A comparison of (45) and (46) leads to a geometric interpretation of the charges. Thus, for the electrical charge e and the charge $\overline{g}$ determining interaction with the Z-boson, we have, respectively,

$$e = \frac{4\sqrt{k_g}}{c^2} \hbar c \frac{\alpha \lambda^5 \sigma^6}{\sqrt{(\lambda^5)^2 + (\sigma^6)^2}}; \quad \overline{g} = \frac{4\sqrt{k_g}}{c^2} \hbar c \alpha \sqrt{(\lambda^5)^2 + (\sigma^6)^2}. \tag{53}$$

From these or other formulae it is possible to determine relations between the constants:

$$\alpha\lambda^5 = \frac{c^2 g_1}{4\sqrt{k_g}\hbar c}; \quad \alpha\sigma^6 = \frac{c^2 g_2}{4\sqrt{k_g}\hbar c}; \quad \alpha^{-1} = 2\sqrt{3}\frac{\hbar\sqrt{k_g}}{ec} \simeq 1,7 \cdot 10^{-31} cm, \tag{54}$$

where we put $\lambda^5 \simeq 1$; $\sigma^6 \simeq \sqrt{3}$; $\lambda^6 = 0$.

4.4. The fermion sector of the 7-dimensional model

In a consideration of fermions in the 7-dimensional theory, it is necessary to elucidate the following issues:

1. **The 7-hedron method**. In 7-dimensional theory, spinors should be described using the 7-hedron method, such that the 7-dimensional metric tensor is presented in the form

$$G_{\mu\nu} = \sum_P G_M(P) G_N(P) = \sum_\alpha g_M(\alpha) g_N(\alpha) - \xi_M \xi_N - \lambda_M \lambda_N - \sigma_M \sigma_N, \tag{55}$$

where

$$G_M(\alpha) = g_M(\alpha); \quad G_M(4) = \xi_M; \quad G_M(5) = \lambda_M; \quad G_M(6) = \sigma_M \tag{56}$$

with the corresponding orthonormality conditions.

2. **Representation of the matrices** Γ_M. In the 7-dimensional theory, the role of the Dirac matrices is played by the 8-row matrices Γ_M. However, by (25) they are coordinate dependent. A direct generalization of the constant Dirac matrices is presented by projections of the matrices Γ_M onto a local set of orthogonal vectors of the 7-hedron $\Gamma(P) = \Gamma_M G^M(P)$. These constant vectors (their coordinate dependence is transferred to the 7-hedron components) are generators of the Clifford algebra $C(1,6)$ [15].

We choose the following representation of the matrices:

$$\Gamma(\alpha) = \Gamma_N g^N(\alpha) = \begin{pmatrix} 0 & \gamma(\alpha) \\ \gamma(\alpha) & 0 \end{pmatrix}; \tag{57}$$

$$\Gamma(4) = i\begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}; \quad \Gamma(5) = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}; \quad \Gamma(6) = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}, \tag{58}$$

where I_4 is the 4-row unit matrix and $\gamma(\alpha) \equiv \gamma^\alpha$ are the constant 4-row Dirac matrices.

3. **The left and right components of spinors**. According to the representation of the $\Gamma(M)$ matrices (57) – (58), the 8-component Ψ-function are naturally split into two 4-component functions:

$$\Psi = \begin{pmatrix} \Psi(1) \\ \Psi(2) \end{pmatrix} \to \overline{\Psi} = \Psi^\dagger \Gamma(0) = (\overline{\Psi}(2), \overline{\Psi}(1)), \tag{59}$$

where $\overline{\Psi}(s) = \Psi^\dagger(s)\gamma(0); \quad s = 1, 2.$

Let us decompose the 4-component functions into the left and right constituents by the standard formulae:

$$\begin{aligned} \Psi_L(s) &= \frac{1}{2}(1 + i\gamma_5)\Psi(s) \to \overline{\Psi}_L(s) = \frac{1}{2}\overline{\Psi}(s)(1 - i\gamma_5); \\ \Psi_R(s) &= \frac{1}{2}(1 - i\gamma_5)\Psi(s) \to \overline{\Psi}_R(s) = \frac{1}{2}\overline{\Psi}(s)(1 + i\gamma_5), \end{aligned} \tag{60}$$

then we have for (59)

$$\Psi = \begin{pmatrix} \Psi_L(1) + \Psi_R(1) \\ \Psi_L(2) + \Psi_R(2) \end{pmatrix}; \quad \overline{\Psi} = (\overline{\Psi}_L(2) + \overline{\Psi}_R(2);\ \overline{\Psi}_L(1) + \overline{\Psi}_R(1)). \tag{61}$$

4. **The dependence of the functions on the additional coordinates.** In the Weinberg-Salam model the left and right components of the leptons have different isotopic properties and different hypercharges. According to Eqs. (22), (47) and (48), we introduce the x^5 and x^6 dependence of the spinor functions

$$\Psi_L(s) = \left(a_{Ls}\nu_L e^{i\beta x^6} + b_{Ls} e_L e^{-i\beta x^6}\right) e^{-i\alpha x^5}; \quad \Psi_R(s) = a_{Rs}\nu_R + b_{Rs} e_R e^{-2i\alpha x^5}, \tag{62}$$

where $s = 1, 2$; a_{Ls}, a_{Rs}, b_{Ls}, b_{Rs} are constant coefficients determined from normalization conditions and correspondence with the standard relations of the Weinberg-Salam model.

The right component of neutrino can be excluded by putting $a_{R1} = a_{R2} = 0$ (it is massless), and the right component of an electron can be considered as isoscalar, so that, in particular, we can put $b_{R1} = 1; b_{R2} = 0$. Then the nonzero left components of the leptons in (62) have the form of a 2-component spinor, consisting of ν_L and e_L, in isotopic space.

The data on the x^5 and x^6 dependences of all the above quantities (in a coordinate frame where $\lambda^6 = 0$) are collected in Table 1.

Particles	x^5 dependence	Hyper-charge Y	x^5 dependence	Isospin projection T_3
			Isodoublet	
ν_L	$\exp(-i\alpha x^5)$	-1	$\exp(i\alpha x^6)$	$+1/2$
e_L	$\exp(-i\alpha x^5)$	-1	$\exp(-i\alpha x^6)$	$-1/2$
			Singlets	
ν_R	$\exp(0)$	0	$\exp(0)$	0
e_R	$\exp(-2i\alpha x^5)$	-2	$\exp(0)$	0
B_α	$\exp(0)$	0	$\exp(0)$	0
			Isotriplet	
W^+_α	$\exp(0)$	0	$\exp(2i\alpha x^6)$	$+1$
W^-_α	$\exp(0)$	0	$\exp(-2i\alpha x^6)$	-1
$A(3)_\alpha$	$\exp(0)$	0	$\exp(0)$	0
			Isodoublet	
φ_0	$\exp(i\alpha x^5)$	1	$\exp(-i\alpha x^6)$	$-1/2$
φ_+	$\exp(i\alpha x^5)$	1	$\exp(i\alpha x^6)$	$+1/2$

4.5. Higgs scalar fields and rest masses

In the 7-dimensional theory the doublet of the Higgs scalar fields arises from the conformal factor χ^2 in

$$\tilde{G}_{MN} = \chi^2 G_{MN}; \qquad \sqrt{\tilde{G}^{(7)}} = \chi^7 \sqrt{G^{(7)}}, \tag{63}$$

where $\tilde{G}_{MN}$ is the initial 7-dimensional metric tensor. In the above consideration we had deal with the resulting metric tensor G_{MN}. It is suggested to put $\chi^2 = -\tilde{G}_{44}$.

Using the known formulae for conformal transformations, we obtain the geometric Lagrangian hyperdensity in the form

$$-\frac{\sqrt{\tilde{G}^{(7)}}}{2\tilde{æ}c}\,{}^7\tilde{R} = -\frac{\sqrt{G^{(7)}}}{2\tilde{æ}c}\chi^5\left({}^7R - 12G^{MN}\frac{\nabla_M\nabla_N\chi}{\chi} - 18G^{MN}\frac{\chi_{,M}\chi_{,N}}{\chi^2}\right). \tag{64}$$

In the present case our point of interest is the terms described the scalar field. After the 4+1+1+1 splitting they have the following form

$$\tilde{\mathcal{L}}_\chi = -\frac{3\sqrt{G^{(7)}}}{\tilde{æ}c}\chi^3\{5g^{\mu\nu}(\partial^{\dagger\dagger}_\mu\chi)\partial^{\dagger\dagger}_\nu\chi +$$

$$+ 2\chi[\partial_4^{\dagger\dagger 2}\chi + \partial_5^{\dagger\dagger 2}\chi + \partial_6^{\dagger\dagger 2}\chi] + 3[(\partial^{\dagger\dagger}_4\chi)^2 + (\partial^{\dagger\dagger}_5\chi)^2 + (\partial^{\dagger\dagger}_6\chi)^2]\} + h.c. \tag{65}$$

According to Eqs. (47) and (48), we introduce the x^5 and x^6 dependence of the conformal factor

$$\chi = 1 + b_0\left(\varphi_0\exp[i\alpha(x^5 - x^6)] - \varphi_0^\star\exp[-i\alpha(x^5 - x^6)]\right) +$$
$$+ b_+\left(\varphi_+\exp[i\alpha(x^5 + x^6)] - \varphi_+^\star\exp[-i\alpha(x^5 + x^6)]\right), \tag{66}$$

where b_0, b_+ are constants, φ_0 и φ_+ are complex scalar fields, which correspond to the neutral and charged components of Higgs doublet. Let us put $b_0 \neq 0$; $b_+ = 0$ in accordance with the special (unitar) gauge of Weinberg-Salam's model.

Substituting (66) in (65) and averaging over compactification periods of extra dimensions, we find the Lagrangian density of the Higgs neutral scalar field:

$$\mathcal{L}_\varphi = \frac{30}{æ}\sqrt{G^{(7)}}b_0^2\left\{g^{\mu\nu}(\partial_\mu^{\dagger\dagger}\varphi_0^\star)\partial_\nu^{\dagger\dagger}\varphi_0 - \alpha^2[(\lambda^5)^2 + (\sigma^6)^2]\varphi_0^\star\varphi_0\right\} + O(b_0^4). \tag{67}$$

The Higgs mechanism corresponds to presentation of the φ_0 in the form $\varphi_0 = \eta_0 + \phi_0$, where η_0 is a constant, ϕ_0 is an effective scalar field.

In 7-dimensional theory, all particles are originally massless. Z- and W-boson rest masses arise due to interactions with the scalar field φ_0 via the mechanism like that of Higgs's. In particular, Z-boson rest mass is

$$m_z c^2 =\simeq 2(\hbar c)(b_0\eta_0)^2\alpha\sqrt{(\lambda^5)^2 + (\sigma^6)^2}. \tag{68}$$

In our work [9] it was shown that the fermion rest masses have another nature. There are two contributions to a fermion mass. First one arises from derivations of the fermion wave functions upon the additional coordinates. The second contribution is conditioned by extra coordinate derivations of the Higgs scalar field contained in the 7-dimensional Ricci rotation coefficients $\Delta_M(NP)$. These contributions are equal each other and have opposite signs. Therefore, to obtain non-zero rest masses we must introduce the x^4 dependence of fermion wave functions. Then the fermion rest masses arise from x^4 derivations of the fermions wave function (22).

5. The 8-dimensional geometrical model of gravi-strong interactions

5.1. Main ideas and methods of the 8-dimensional model

To construct a multidimensional geometrical model of gravi-strong interactions, we should solve the following problems:

1. It is necessary to describe 3 kinds of colour charges of chromodynamics [16, 17] by geometric methods.

2. As in chromodynamics, the strong interactions are transfered by 8 kinds of gluons, it is necessary to show the geometric image of these physical vector fields in a multidimensional geometric model.

3. The gauge group $SU(3)$ leads to essentially nonlinear expressions in the boson sector of the Lagrangian. We should show that all these nonlinear terms can be described in the frames of a multidimensional geometric model of Kaluza-Klein type.

4. We should demonstrate also that it is possible to describe the interaction of fermions with gluons in accordance with the fermion sector of chromodynamics [9, 18].

To solve the above problems, the following ideas and methods were used:

(i) As has been previously shown [19], the 7th dimension is insufficient for solving the above problems. It was suggested to use an 8-dimensional geometric model with the signature $(+---|----)$. The main reason for introducing three additional dimensions (in addition to the four classical coordinates and x^4) is the necessity to describe three colour charges (to solve the first problem from the above list). The charges in Kaluza-Klein theory are known to correspond to additional momentum components. Three charges are the three new dimensions (of momentum). We denote these three additional coordinates as x^7, x^8, x^9 taking into account that all previous numbers are occupied to describe the classical space-time and electro-weak interactions.

(ii) Three additional coordinates are chosen to be compact. It is suggested to use 3-torus topology. It means all fields possessing colour charges should depend on the additional coordinates in a cyclic manner. To describe the three colour states of quarks, it is suggested to use their following dependence on the additional coordinates:

$$q_1 \sim exp(i\gamma x^7); \quad q_2 \sim exp(i\gamma x^8); \quad q_3 \sim exp(i\gamma x^9); \tag{69}$$

where γ is some new constant determining a compactification radius of additional dimensions characterizing the strong interactions. Due to the symmetry of all three charges in chromodynamics, these constants are chosen equal for all dimensions.

(iii) To describe the 8 gluons, it is suggested to use a metric version of an 8-dimensional theory, where all gluons are described by the multidimensional metric, as is the case in the 7-dimensional model of gravi-electroweak interactions.

Two of 8 gluons are known to be neutral in the sense of colour and six ones are charged. According to (69), the charged gluons should have the following dependence on the additional coordinates:

$$X^{\pm}_{\mu} \sim \exp[\mp i\gamma(x^7 - x^8)]; \quad Y^{\pm}_{\mu} \sim \exp[\mp i\gamma(x^7 - x^9)]; \quad Z^{\pm}_{\mu} \sim \exp[\mp i\gamma(x^8 - x^9)]. \tag{70}$$

(iv) In the standard approach to the description of spinors using Clifford's algebra over the field of real numbers there is a close relation between the dimension and signature of the manifold and the number of spinor components. For the dimension 8 we should use 16-component spinors.

(v) In accordance with the spirit of general relativity, the key expression is chosen as an 8-dimensional Lagrangian hyperdensity consisting of a geometric part and a spinor matter contribution:

$$\mathcal{L}_{(8)} = \sqrt{-G^{(8)}}\left[\frac{-{}^8R}{2\tilde{æ}c} + \frac{i\hbar c}{2}\overline{\Psi}\Gamma^M\nabla_M\Psi + (h.c.)\right], \tag{71}$$

where $G^{(8)} = det(G^8)$, ${}^8R$ is the 8-dimensional scalar curvature; Γ_M are 16×16 Dirac matrices which satisfy (25) with the index $M = 0, 1, 2, 3, 4, 7, 8, 9$.

(vi) As in the other papers [7, 8, 19], it is suggested to use a generalization of the monad method for 4 additional coordinates. In this case it is the tetrad method of 4+1+1+1+1 splitting.

(vii) To obtain the ultimate formulae, the 8-dimensional expression must be averaged (integrated) over the additional coordinates.

5.2. The metric and the physico-geometric tensors

Using the ideas of the monad, dyad and triad methods, let us expound the necessary information on the tetrad method of 4+1+1+1+1 splitting. The 8-dimensional metric tensor in the above chosen signature has the following form in the tetrad method:

$$G^{(8)}_{MN} = g_{MN} - \xi_M\xi_N - \zeta_M\zeta_N - \eta_M\eta_N - \omega_M\omega_N, \tag{72}$$

where $\xi_M, \zeta_M, \eta_M, \omega_M$ are the four 8-dimensional tetradic vectors that satisfy to orthonormality conditions. We will use a gauge like the 4-chronometric one in the general relativity, when the tetradic vector components depend on the components of the 8-dimensional metric tensor in the following way:

$$\xi^M = \frac{G^M_4}{\sqrt{-G_{44}}}; \quad \omega^M = \frac{\hat{G}^M_9}{\sqrt{-\hat{G}_{99}}}; \quad \eta^M = \frac{\tilde{G}^M_8}{\sqrt{-\tilde{G}_{88}}}; \quad \zeta^M = \frac{\tilde{\tilde{G}}^M_7}{\sqrt{-\tilde{\tilde{G}}_{77}}}, \tag{73}$$

where

$$\hat{G}_{MN} = {}^8G_{MN} + \xi_M\xi_N; \quad \tilde{G}_{MN} = \hat{G}_{MN} + \omega_M\omega_N; \quad \tilde{\tilde{G}}_{MN} = \tilde{G}_{MN} + \eta_M\eta_N. \tag{74}$$

The metric tensor components have much involved expressions in the tetrad method, therefore we will not write them out here.

By analogy with the 7-dimensional geometric model of gravi-electroweak interactions, we will suppose that the 4-dimensional components of the tetrad vectors depend in a cyclic manner on the additional coordinates x^7, x^8, x^9 and do not depend on x^4:

$$\xi_\alpha = C_0\{a_4A_\alpha + b_4B_\alpha + x_4^+X_\alpha^+\exp[-i\beta(x^7 - x^8)] + x_4^-X_\alpha^-\exp[i\beta(x^7 - x^8)] +$$
$$+ y_4^+Y_\alpha^+\exp[-i\beta(x^7 - x^9)] + y_4^-Y_\alpha^-\exp[i\beta(x^7 - x^9)] +$$
$$+ z_4^+Z_\alpha^+\exp[-i\beta(x^8 - x^9)] + z_4^-Z_\alpha^-\exp[i\beta(x^8 - x^9)]\}. \tag{75}$$

Here C_0 is some dimensional constant and a_s, b_s, $x_s^\pm$, $y_s^\pm$, $z_s^\pm$ are dimensionless constants which should be determined from the correspondence with the bosonic sector of chromodynamics. The vector fields A_α, B_α, $X_\alpha^\pm$, $Y_\alpha^\pm$, $Z_\alpha^\pm$ represent the 8 gluon fields of standard chromodynamics. The other tetrad vectors $\zeta_\alpha, \eta_\alpha, \omega_\alpha$ have the same form (75) with the difference that the lower index of the constants $a_4, b_4, x_4^\pm, y_4^\pm, z_4^\pm$ shuld be replaced by 7, 8, 9 for $\zeta_\alpha, \eta_\alpha, \omega_\alpha$, respectively.

In addition, we will be restricted to the case when the non-diagonal "scalar" components of the tetrad vectors are zero:

$$\xi^9 = \xi^8 = \xi^7 = \zeta^8 = \zeta^9 = \eta^9 = 0. \tag{76}$$

Therefore due to the symmetry it is natural to put

$$\xi^4 = \zeta^7 = \eta^8 = \omega^9. \tag{77}$$

Using the tetrad method, a number of physico-geometric tensors are built from the components of the metric tensor and their first-order derivatives. Let us write down four tensors only, which correspond (but not directly) to the antisymmetric gluon fields tensors in chromodynamics:

$$F_{\alpha\beta}^{(4)} = \frac{1}{2}[(\xi_{\alpha,\beta} - \xi_{\beta,\alpha}) + \zeta^7(\zeta_\beta\xi_{\alpha,7} - \zeta_\alpha\xi_{\beta,7}) + \eta^8(\eta_\beta\xi_{\alpha,8} - \eta_\alpha\xi_{\beta,8}) + \omega^9(\omega_\beta\zeta_{\alpha,9} - \omega_\alpha\zeta_{\beta,9})]; \tag{78}$$

$$F_{\alpha\beta}^{(7)} = \frac{1}{2}[(\zeta_{\alpha,\beta} - \zeta_{\beta,\alpha}) + \zeta^7(\zeta_\beta\zeta_{\alpha,7} - \zeta_\alpha\zeta_{\beta,7}) + \eta^8(\eta_\beta\zeta_{\alpha,8} - \eta_\alpha\zeta_{\beta,8}) + \omega^9(\omega_\beta\zeta_{\alpha,9} - \omega_\alpha\zeta_{\beta,9})]; \tag{79}$$

$$F_{\alpha\beta}^{(8)} = \frac{1}{2}[(\eta_{\alpha,\beta} - \eta_{\beta,\alpha}) + \zeta^7(\zeta_\beta\eta_{\alpha,7} - \zeta_\alpha\eta_{\beta,7}) + \eta^8(\eta_\beta\eta_{\alpha,8} - \eta_\alpha\eta_{\beta,8}) + \omega^9(\omega_\beta\eta_{\alpha,9} - \omega_\alpha\eta_{\beta,9})]; \tag{80}$$

$$F_{\alpha\beta}^{(9)} = \frac{1}{2}[(\omega_{\alpha,\beta} - \omega_{\beta,\alpha}) + \zeta^7(\zeta_\beta\omega_{\alpha,7} - \zeta_\alpha\omega_{\beta,7}) + \eta^8(\eta_\beta\omega_{\alpha,8} - \eta_\alpha\omega_{\beta,8}) + \omega^9(\omega_\beta\omega_{\alpha,9} - \omega_\alpha\omega_{\beta,9})], \tag{81}$$

where there is no dependence on x^4 because the cylindricity condition with respect to x^4 is used.

For our purpose, of greatest interest is the 8-dimensional scalar curvature ${}^8R$ contained in the Lagrangian hyperdensity (71). After the 4+1+1+1+1 splitting it has the following form:

$${}^8R = {}^4R + F_{\alpha\beta}^{(4)}F^{(4)\alpha\beta} + F_{\alpha\beta}^{(7)}F^{(7)\alpha\beta} + F_{\alpha\beta}^{(8)}F^{(8)\alpha\beta} + F_{\alpha\beta}^{(9)}F^{(9)\alpha\beta} + \ldots, \tag{82}$$

where the dots replace other physico-geometric tensors describing the mass terms.

The boson sector of the 8-dimensional geometric model is described by the first part of the Lagrangian hyperdensity (71), where the expression (82) for the 8-dimensional scalar curvature is used. The 4-dimensional scalar curvature ${}^4R$ obviously describes the gravity, and the remaining terms in (82) conform to the Lagrangian of vector boson fields in standard chromodynamics. Below we will assume that the gravitational contribution is negligibly small and discuss only the gluon contributions in the actually flat 4-dimensional space-time. Introducing in (82) the representation (75) of the tetrad components in the form of physical gluon fields and exponential terms, we get, after averaging with respect to the additional coordinates, a 4-dimensional geometric Lagrangian density. The expression is very long and we will not write it down here.

5.3. The boson sector of 8-dimensional geometric model

We must show that it is possible to choose the coefficients from (75) in such a way that the boson geometric part will be equal to the boson part of standard chromodynamics.

Comparing the density of geometric Lagrangian density of the 8-dimensional model with boson part of the correspondent density in standard chromodynamics, we obtain 46 conditions (equations) for the 33 coefficients of resolution of tetrad vector fields of gluons and for one constant $C = \beta\zeta^7 C_0/g_0$, where g_0 is the constant of the $SU(3)$ group in chromodynamics. It should be noted that the set of equations that has been obtained is not overdetermined. Solving it, we find:

$$a_4 = 0; \quad a_7 = -a_8 = \pm\frac{1}{\sqrt{2}}; \quad a_9 = 0; \tag{83}$$

$$b_4 = 0; \quad b_7 = b_8 = \pm\frac{1}{\sqrt{6}}; \quad b_9 = \mp\sqrt{\frac{2}{3}}; \quad C = \pm\frac{1}{\sqrt{2}}. \tag{84}$$

In all these expressions the plus-minus signs correspond to one another. The point is that these coefficients are contained in a number of expressions squared. It is of interest to note that the solutions for two triplets of coefficients numbered 7, 8, 9 conform to two (diagonal) Gell-Mann's matrices λ_3 and λ_8.

Substituting these solutions into the remaining equations, we find solutions for the "charged" coefficients. Let us write out the coefficients with the indices "4" separately:

$$x_4^+ = x_4^- = y_4^+ = y_4^- = z_4^+ = z_4^- = \pm 1. \tag{85}$$

It is convenient to represent the remaining coefficients with indices "7, 8, 9" in a 3-dimensional vector form: $\vec{x}^\pm = (x_7^\pm, x_8^\pm, x_9^\pm)$. 8 pairs of solutions are found for them. We present them in the form of a Table 2:

	Variant 1	Variant 2	Variant 3	Variant 4
$\vec{x}^-$	$(0,\pm1,0)$	$(\pm1,0,0)$	$(0,\pm1,0)$	$(\pm1,0,0)$
$\vec{x}^+$	$(\pm1,0,0)$	$(0,\pm1,0)$	$(\pm1,0,0)$	$(0,\pm1,0)$
$\vec{y}^-$	$(0,0,\pm1)$	$(\pm1,0,0)$	$(0,0,\mp1)$	$(\mp1,0,0)$
$\vec{y}^+$	$(\pm1,0,0)$	$(0,0,\pm1)$	$(\mp1,0,0)$	$(0,0,\mp1)$
$\vec{z}^-$	$(0,0,\pm1)$	$(0,\pm1,0)$	$(0,0,\mp1)$	$(0,\mp1,0)$
$\vec{z}^+$	$(0,\pm1,0)$	$(0,0,\pm1)$	$(0,\mp1,0)$	$(0,0,\mp1)$
	Variant 5	Variant 6	Variant 7	Variant 8
$\vec{x}^-$	$(0,\mp1,0)$	$(\mp1,0,0)$	$(0,\mp1,0)$	$(\mp1,0,0)$
$\vec{x}^+$	$(\mp1,0,0)$	$(0,\mp1,0)$	$(\mp1,0,0)$	$(0,\mp1,0)$
$\vec{y}^-$	$(0,0,\pm1)$	$(\pm1,0,0)$	$(0,0,\mp1)$	$(\mp1,0,0)$
$\vec{y}^+$	$(\pm1,0,0)$	$(0,0,\pm1)$	$(\mp1,0,0)$	$(0,0,\mp1)$
$\vec{z}^-$	$(0,0,\mp1)$	$(0,\mp1,0)$	$(0,0,\pm1)$	$(0,\pm1,0)$
$\vec{z}^+$	$(0,\mp1,0)$	$(0,0,\mp1)$	$(0,\pm1,0)$	$(0,0,\pm1)$

The totality of this solutions is doubled due to the two signs before the coefficients with index "4" in (85). It will be shown later on that the conditions following from correspondence of fermion sectors of two theories single out the first variant of solutions. The form of this solution conforms to the structure of three pairs of Gell-Mann's non-diagonal matrices in the Cartan-Weil basis if we construct 3×3 matrices from the coefficients $\overrightarrow{x}^{\pm}$, $\overrightarrow{y}^{\pm}$, $\overrightarrow{z}^{\pm}$ in a special manner.

Let us write out the tetrad components (75) for the first variant of the solutions:

$$\xi_\alpha = C_0[X^+_\alpha \exp[-i\beta(x^7 - x^8)] + X^-_\alpha \exp[i\beta(x^7 - x^8)] + Y^+_\alpha \exp[-i\beta(x^7 - x^9)] +$$

$$Y^-_\alpha \exp[i\beta(x^7 - x^9)] + Z^+_\alpha \exp[-i\beta(x^8 - x^9)] + Z^-_\alpha \exp[i\beta(x^8 - x^9)]]; \quad (86)$$

$$\zeta_\alpha = C_0[\frac{1}{\sqrt{2}}A_\alpha + \frac{1}{\sqrt{6}}B_\alpha + X^+_\alpha \exp[-i\beta(x^7 - x^8)] + Y^+_\alpha \exp[-i\beta(x^7 - x^9)]]; \quad (87)$$

$$\eta_\alpha = C_0[-\frac{1}{\sqrt{2}}A_\alpha + \frac{1}{\sqrt{6}}B_\alpha + X^-_\alpha \exp[i\beta(x^7 - x^8)] + Z^+_\alpha \exp[-i\beta(x^8 - x^9)]]; \quad (88)$$

$$\omega_\alpha = C_0\left[-\frac{2}{\sqrt{6}}B_\alpha + Y^-_\alpha \exp[i\beta(x^7 - x^9)] + Z^-_\alpha \exp[i\beta(x^8 - x^9)]\right]. \quad (89)$$

In the earlier version of a 7-dimensional model of gravi-strong interactions [19] the analogous problem of finding the coefficients in the triad components (75) was solved. The same solutions (83) – (84) for the "neutral" coefficients were found and for the "charged" ones other 8 pairs of solutions were obtained.

5.4. The differential operators and the neutral vector fields

In the present geometrical model a basic role is played by tetrad differential operators:

$$\partial^\star_4 = \xi^M \frac{\partial}{\partial x^M} \Rightarrow \xi^4 \frac{\partial}{\partial x^4}; \quad (90)$$

$$\partial^\star_7 = \zeta^M \frac{\partial}{\partial x^M} \Rightarrow \zeta^7 \frac{\partial}{\partial x^7}; \quad (91)$$

$$\partial^\star_8 = \eta^M \frac{\partial}{\partial x^M} \Rightarrow \eta^8 \frac{\partial}{\partial x^8}; \quad (92)$$

$$\partial^\star_9 = \omega^M \frac{\partial}{\partial x^M} \Rightarrow \omega^9 \frac{\partial}{\partial x^9}; \quad (93)$$

$$\partial^\star_\mu = g^M_\mu \frac{\partial}{\partial x^M} = \frac{\partial}{\partial x^\mu} + \xi_\mu \partial^\star_4 + \zeta_\mu \partial^\star_7 + \eta_\mu \partial^\star_8 + \omega_\mu \partial^\star_9. \quad (94)$$

The action of all these operators depends neither on the rang nor on the covariance of quantities to be differentiated. The last operator (94) conforms to the long derivative in chromodynamics. Assume that this operator acts to an arbitrary function Ψ with the following dependence on the additional coordinates:

$$\Psi = \psi(x^\alpha) \exp[i\beta x^4 + i\gamma(\varepsilon_7 x^7 + \varepsilon_8 x^8 + \varepsilon_9 x^9)], \quad (95)$$

where $\psi(x^\alpha)$ depends only on the four classical coordinates. Using the solutions (83) – (84) for the coefficients before neutral vector fields, the following expression for the long derivative is obtained:

$$\partial^\star_\mu \Psi = \left[\frac{\partial}{\partial x^\mu} + \sqrt{2} i\tilde{\gamma} C_o \frac{\varepsilon_7 - \varepsilon_8}{2} A_\mu + \sqrt{2} i\tilde{\gamma} C_o \frac{\varepsilon_7 + \varepsilon_8 - 2\varepsilon_9}{2\sqrt{3}} B_\mu + \cdots\right]\Psi. \tag{96}$$

Here the upper signs in the solutions (83) – (84) are chosen; $\tilde{\gamma} = \gamma\eta^7$. Comparing the long derivative of standard chromodynamics with that in (96) in the 8-dimensional model, one gets the following relations:

$$Q_a = \frac{1}{2}(\varepsilon_7 - \varepsilon_8); \quad Q_b = \frac{1}{2\sqrt{3}}(\varepsilon_7 + \varepsilon_8 - 2\varepsilon_9). \tag{97}$$

Using these formulas and above ones (69), (70) for the dependence on the additional coordinates, let us write out the harmonics and the charges for all the relevant particles in the form of a Table 3:

Particles		ε_7	ε_8	ε_9	Q_a	Q_b	Q_c
Quarks	$q_1 \equiv q_R$	1	0	0	1/2	$1/2\sqrt{3}$	1/3
	$q_2 \equiv q_Y$	0	1	0	−1/2	$1/2\sqrt{3}$	1/3
	$q_3 \equiv q_G$	0	0	1	0	$-1/\sqrt{3}$	1/3
Gluons	A_μ, B_μ	0	0	0	0	0	0
	X^+_μ	−1	1	0	−1	0	0
	Y^+_μ	−1	0	1	−1/2	$-\sqrt{3}/2$	0
	Z^+_μ	0	−1	1	1/2	$0 - \sqrt{3}/2$	0

All charges are given in the units g_0.

Finally, let us show the physical meaning of the differential operators (91) – (93). The eigenvalues of three combinations of these operators represent the above defined values of charges multiplied by some coefficients:

$$\frac{1}{2}(\partial^\star_7 - \partial^\star_8)\Psi = i\tilde{\gamma} Q_a \Psi; \quad \frac{1}{2\sqrt{3}}(\partial^\star_7 + \partial^\star_8 - 2\partial^\star_9)\Psi = i\tilde{\gamma} Q_b \Psi; \quad \frac{1}{3}(\partial^\star_7 + \partial^\star_8 + \partial^\star_9)\Psi = i\tilde{\gamma} Q_c \Psi. \tag{98}$$

The rest differential operator (90) should interpret as mass one.

5.5. The fermion sector of the 8-dimensional model

Let us turn to the discussion of the fermion sector in the 8-dimensional model of gravi-strong interactions. If we neglect the gravitational interaction and exlude the mass terms, the non-geometrical part of the Lagrangian hyperdensity takes the following form:

$$\mathcal{L}_{Ferm} = \frac{\sqrt{-g}}{2}(i\hbar c \overline{\Psi} \tilde{\Gamma}^\mu \partial^*_\mu \Psi + \ldots) + (h.c.), \tag{99}$$

where the long derivative operator is written in (94) and the 16×16 matrices generalizing the Dirac matrices are projected onto the 4-dimensional direction: $\tilde{\Gamma}^\mu = \Gamma^M g^\mu_M$.

Choose for the matrices $\tilde{\Gamma}^\mu$ following representation in terms of the standard 4×4 Dirac matrices γ^μ (in flat space-time):

$$\tilde{\Gamma}^\mu = \begin{pmatrix} 0 & 0 & 0 & \gamma^\mu \\ 0 & 0 & \gamma^\mu & 0 \\ 0 & \gamma^\mu & 0 & 0 \\ \gamma^\mu & 0 & 0 & 0 \end{pmatrix}. \tag{100}$$

In the 8-dimensional manifold with these $\tilde{\Gamma}$ -matrices we will use the 16-component complex spinors:

$$\Psi = \begin{pmatrix} \Psi(1) \\ \Psi(2) \\ \Psi(3) \\ \Psi(4) \end{pmatrix} \Rightarrow \overline{\Psi} = \Psi^{\dagger}\Gamma(0) = (\overline{\Psi}(4), \overline{\Psi}(3), \overline{\Psi}(2), \overline{\Psi}(1)), \tag{101}$$

where $\Psi(s)$ and $\overline{\Psi}(s) = \Psi^{\dagger}(s)\gamma(0)$ are 4-component complex spinors ($s = 1, 2, 3, 4$).

Taking into account the symmetry of the three quarks, it is suggestive to use the following representation of the above spinors in terms of the quark wave functions:

$$\Psi(s) = c_s[q_1 \exp(i\beta x^7) + q_2 \exp(i\beta x^8) + q_3 \exp(i\beta x^9)], \tag{102}$$

where, in accordance with (69), for three quark colours a cyclic dependence on the additional coordinates is introduced; q_i are the parts of quark wave functions depending only on the 4 coordinates, c_s are the four complex coefficients to be determined from the conformity to standard chromodynamics.

Substituting (101), (102) into (99) and averaging with respect to periods of the additional coordinates, one obtains a 4-dimensional form of the fermion part of Lagrangian density. Comparing it with the fermion sector of chromodynamics, one obtains a set of 9 independent algebraic relations between the constants c_s and the coefficients of the boson sector $a_s, b_s, x_s^{\pm}, y_s^{\pm}, z_s^{\pm}$. Assuming the coefficients c_s to be equal, we obtain these conditions in a form:

$$a_7 = \frac{1}{2C}; \quad a_8 = -\frac{1}{2C}; \quad a_9 = 0; \quad b_7 = \frac{1}{2\sqrt{3}C}; \quad b_8 = \frac{1}{2\sqrt{3}C}; \quad b_9 = -\frac{1}{\sqrt{3}C}; \tag{103}$$

$$x_7^+ + x_8^- = y_7^+ + y_9^- = z_8^+ + z_9^- = \frac{\sqrt{2}}{C} \tag{104}$$

There is one more condition for the coefficients c_s

$$c_1^* c_1 + c_2^* c_2 + c_3^* c_3 + c_4^* c_4 = 1. \tag{105}$$

Using these relations in the first variant of the solutions for the coefficients a_s, b_s, $x_s^{\pm}$, $y_s^{\pm}$, $z_s^{\pm}$ written in Table 2, we find that the relations (103) – (104) are satisfied identically. The condition (105) is satisfied if we put

$$c_1 = c_2 = c_3 = c_4 = \frac{1}{2}. \tag{106}$$

It should be noted that three more variants from Table 2 satisfy the above conditions unless we put the coefficients c_s to be equal.

Conclusion

Finishing the paper, we make some conclusions and remarks:

1. Based on Kaluza-Klein's ideas, one manages to develop concrete versions of multidimensional geometrical models unifying general relativity with theory of other interactions, in particular:

(1) 5- and 6-dimensional geometrical models of the unified theory of gravitation and electromagnetism developing and generalizing Kaluza and Klein-Rumer's versions of 5-dimensional theories;

(2) 7-dimensional version of the unified theory of gravitational and electroweak interactions combining the patterns of Einstein's general relativity and Weinberg-Salam's electroweak interaction model (the latter is well embedded in the 7-dimensional theory);

(3) 8-dimensional geometrical model of gravitational and strong interactions unifying General Relativity with classical chromodynamics.

2. A futher increase in dimensions up to ten allows one to construct the unified geometrical theory of gravi-electroweak and strong interactions.

3. There are the set of the principal problems of Kaluza-Klein's classical models, among them including

(i) the additional dimension compactification problem;

(ii) the problem of number of using harmonics in the cyclic dependence of all quantities on the additional coordinates;

(iii) the dimension number problem for multidimensional models;

(iv) the problem of physical meaning of solutions of Einstein's multidimensional equations.

In author's opinion, these and other problems should be solved beyond the geometrical paradigm.

4. In our works [9, 20] it was developed another approach to the physical interacting, named binary geometrophysics. The physical basis of this approach comprises ideas of three kinds:

1) a macroscopic nature of the classical space-time,

2) a direct interparticle action (Fokker-Feynman's action-at-a-distance concept, an alternative to field theory),

3) Kaluza-Klein's type multidimensional geometrical models of physical interactions.

A prototype of the multidimensional metric is derived from binary geometrophysics notions, which justifies Kaluza-Klein's type multidimensional geometrical models of physical interactions and, in particular, the idea of additional dimensions in the microworld manifesting themselves as electroweak and strong interactions.

A thorough analysis of multidimensional geometrical models from binary geometrophysics viewpoint permitted us to revise the principal difficulties of the Kaluza-Klein's classical models.

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