ADELIC FORMULAE FOR STRING AMPLITUDES IN FIELDS OF ALGEBRAIC NUMBERS

V. S. Vladimirov
Steklov Mathematical Institute, Moscow, Russia

1. Introduction

In recent years p-adic numbers and p-adic analysis have begun to be applied to various models in mathematical physics (see books Vladimirov et al [1] and Khrennikov [2] and references therein). The physical motivation for that are the following arguments.

In quantum gravitation and in string theory it was established the lower bound

$$\Delta x \geq l_P \tag{1.1.}$$

for an uncertainty Δx in length measurement. The fundamental constant

$$l_P = (G\bar{h}/c^3)^{1/2},$$

called *the Planck length*, is very small, being equal to about $10^{-33}cm$.

The inequality (1.1.) provides a non-Archimedean structure of the physical space (and time) at the Planck distances, so it is impossible to use the real numbers with its Archimedean structure as the space-time coordinates.

Thus the field of real numbers $\mathbb{R}$ has to be changed by another number field. In any case such new field should contain the field of rational numbers $\mathbb{Q}$, as integer numbers (and thus rational ones) form a base of count of beings in any inhabitable reasonable world. These arguments lead to the following conclusion: in order to get a desired number field it is necessary to assign in the field $\mathbb{Q}$ a new, non-Archimedean, metric (topology), and to close $\mathbb{Q}$ with respect to this metric.

A question arises how to choose such new metric? Mathematics has given an answer to this question. In 1899 or close to the German mathematician Hensel discovered new norms on the field $\mathbb{Q}$ different from Euclidean one. These (nonequivalent) norms are indexed by the prime numbers $p = 2, 3, 5, \ldots, 137, \ldots$; they are called *p-adic*, and denoted by $|.|_p$.

The norm $|.|_p$ is defined by the following way. Any rational number $x \neq 0$ can be represented uniquely in the form

$$x = \pm p^{\gamma}(x_{\gamma} + x_{\gamma+1}p + \ldots + x_k p^{k-\gamma}) \tag{2.1.}$$

for some $\gamma, k \in Z, k \geq \gamma$, and $x_j = 0, 1, \ldots, p-1$, $j = \gamma, \gamma+1, \ldots, k, x_{\gamma} \neq 0, x_k \neq 0$ depending on x and p. (Here Z is the ring of rational integer numbers.) By definition, p-adic norm of x is

$$|x|_p = p^{-\gamma}, \quad |0|_p = 0.$$

This norm possesses the standard properties of norms:

$$(i) \quad |x|_p \geq 0, \quad |x|_p = 0 \Longleftrightarrow x = 0,$$

$$(ii) \quad |xy|_p = |x|_p |y|_p,$$

however the triangle inequality is fulfilled in the stronger form

$$(iii) \quad |x + y|_p \leq \max(|x|_p, |y|_p)\,.$$

A norm satisfying (iii) is called *non-Archimedean*, so the p-adic norm $|.|_p$ is non-Archimedean one; it takes only discrete values $p^{-\gamma}$, $\gamma \in Z$.

The remarkable Ostrovskii theorem (see, e.g., [3]) states that in the field $\mathbb{Q}$ there exists no non-equivalent norms different from Euclidean and p-adic ones. Completion of the field $\mathbb{Q}$ with respect to the norm $|.|_p$ forms *the field of p-adic numbers*, which is denoted by $\mathbb{Q}_p$. The fields $\mathbb{Q}_p, \quad p = 2, 3, 5, \ldots$ are similar to the field $\mathbb{R} = \mathbb{Q}_\infty$ which is the completion of $\mathbb{Q}$ with respect to the Euclidean norm $|.| = |.|_\infty$. Thus owing to (2.1.) when $k \to \infty$ every point $x \in \mathbb{Q}_p$, $|x|_p = p^{-\gamma}$ can be uniquely represented in the following canonical form

$$x = p^\gamma (x_\gamma + x_{\gamma+1} p + \ldots), \quad x_j = 0, 1, \ldots, p-1, \ x_\gamma \neq 0, \tag{3.1.}$$

and also the series (3.1.) converges in the norm $|.|_p$.

The field $\mathbb{Q}_p$ is a complete locally compact totally disconnected space with topological dimension 0. It obeys unusual geometrical properties, namely: disc $B_\gamma = [x \in \mathbb{Q}_p : |x|_p \leq p^\gamma]$ as well as circomference $S_\gamma = [x \in \mathbb{Q}_p : |x|_p = p^\gamma]$ are open and closed (clopen) sets without boundaries, every point of a disc is its center, if two discs have a common point then one of them is contained in another, any open set is an union of disjoint discs, all triangles are isosceles, and so on.

Rational numbers are densely contained in all fields $\mathbb{Q}_p$. Moreover, there are no other numbers contained in all fields $\mathbb{Q}_p$. A specific role of rational numbers appears in the so-called *adelic* formula

$$|x-y|_\infty \prod_{p=2}^{\infty} |x-y|_p = 1, \quad x, y \in \mathbb{Q}, \ x \neq y, \tag{4.1.}$$

which connects the Euclidean distance $|x-y|_\infty$ between rational points x and y in the real space $\mathbb{R}$ with the corresponding distances $|x-y|_p$ in all p-adic spaces $\mathbb{Q}_p, \quad \infty, 2, 3, 5, \ldots$. So owing to the equality (4.1.) the measurement of interval $x - y$ with rational ends $x, y \in \mathbb{Q}$ in $\mathbb{R}$ is equivalent to its measurements in all $\mathbb{Q}_p, \quad p = 2, 3, 5, \ldots$.

This result still confirms the statement that fundamental physical theory should base on rational numbers either with the p-adic metric (for the Planck distances) or with the Euclidean metric (for macroscopic distances). A hypothesis on possible non-Archimedean p-adic structure of space-time at the Planck distances has been suggested by Volovich [4–6] in 1987 (see also some discussions [7] on relativity of measurement of time in the Poincaré spirit [8]).

It should be pointed out that for construction of a logically complete theory of space-time one needs use ideal (irrational) numbers belonging to completions of rational numbers with respect to p-adic metric, i.e. to the space $\mathbb{Q}_p$.

However, fundamental (*physically observable*) elements of space $\mathbb{Q}_p$ are rational numbers $x =$ (2.1.) considered as results of some measurements. The quantity $p^\gamma = |x|_p^{-1}$ is *the exactness* of x-measurement (this means that we are sure only in the digit x_γ but the previous digits $x_{\gamma-1}$ in (2.1.) are not well defined). The adelic formula (4.1.) (for $y = 0$) shows that absolute value of x, $|x| = |x|_\infty$, is equal to the product on p of all exactnesses p^γ.

Example [9]. The sum of series

$$\sum_{n=0}^{\infty} n!$$

belongs to all $\mathbb{Q}_p$. Question: is it rational number? Nevertheless

$$\sum_{n=0}^{\infty} n! n = -1.$$

In this report we expose some results on adelic formulae for the fourpoint crossingsymmetric tree string Veneziano amplitudes (open strings) and Virasoro amplitudes (closed strings) in any field of algebraic numbers and for any (ramified or non) quasicharacters*.

In the following we shall denote: by $K^\times$ the group of invertible elements of a ring (field) K; by $A_K(A_K^\times)$ the ring (group) of adeles (ideles) if K is a field of algebraic numbers.

2. Adelic Formulae for String Amplitudes

In 1987 Freund and Witten [10] and, independently a little later, Volovich [11] suggested the adelic formula on group of ideles $A_{\mathbb{Q}}^\times$

$$V(s,t,u)\prod_{p=2}^{\infty} V_p(s,t,u) = 1, \quad s+t+u=-8, \tag{1.2.}$$

for the fourpoint crossingsymmetric Veneziano amplitudes (for open string)

$$V(s,t,u) = B_\infty(-\alpha(s),-\alpha(t),-\alpha(u)), \quad \alpha(s) = 1+s/2, \tag{2.2.}$$

$$V_p(s,t,u) = B_p(-\alpha(s),-\alpha(t),-\alpha(u)), \quad p=2,3,\ldots, \tag{3.2.}$$

where (s,t,u) are the Mandelstam variables;

$$B_p(\alpha,\beta,\gamma) = \Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma), \quad p=\infty,2,3,\ldots, \quad \alpha+\beta+\gamma=1 \tag{4.2.}$$

are beta-functions of fields $\mathbb{R}=\mathbb{Q}_\infty$ and $\mathbb{Q}_p$ for unramified quasicharacters $|x|_p^\alpha$ and $|y|_p^\beta$ respectively;

$$\Gamma_\infty(\alpha) = 2(2\pi)^{-\alpha}\Gamma(\alpha)\cos\pi\alpha/2, \tag{5.2.}$$

$$\Gamma_p(\alpha) = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}, \quad p=2,3,\ldots \tag{6.2.}$$

are corresponding gamma-functions; $\Gamma(\alpha)$ is the classical Euler gamma-function.

Similar adelic formula on the group of ideles of the Gaussian field $\mathbb{Q}(\sqrt{-1})$

$$WW_2\prod_{p\equiv 1(4)} W_p^2 \prod_{p\equiv 3(4)} W_{p^2}(s,t.u) = 2, \quad s+t+u=-32 \tag{7.2.}$$

was suggested also in [10] for the fourpoint crossingsymmetric Virasoro amplitudes (for closed strings)

$$W(s,t,u) = B_\omega(-\alpha(s),-\alpha(t),-\alpha(\gamma)), \quad \alpha(s) = 1+s/8, \tag{8.2.}$$

$$W_q(s,t,u) = B_q(-\alpha(s),-\alpha(t),-\alpha(\gamma)), \quad p=2,3,\ldots, \tag{9.2.}$$

where

$$B_q(\alpha,\beta,\gamma) = \Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma), \quad q=\omega, q(2)=2,$$

$$q(p)=p, p\equiv 1(\mathrm{mod}\,4), q(p)=p^2, p\equiv 3(\mathrm{mod}\,4), \quad \alpha+\beta+\gamma=1 \tag{10.2.}$$

are beta-functions of local fields $\mathbb{C}=\mathbb{Q}_\omega$, $\mathbb{Q}_2(q=2)$, $\mathbb{Q}_p, p\equiv 1(\mathrm{mod}\,4)(q=p)$ and $\mathbb{Q}_p(\sqrt{-1}), p\equiv 3(\mathrm{mod}\,4)$ $(q=p^2)$;

$$\Gamma_\omega(\alpha) = 2(2\pi)^{-2\alpha}\Gamma^2(\alpha)\sin\pi\alpha \tag{11.2.}$$

is gamma-function for unramified quasicharacter $|x\bar{x}|^\alpha$ of field $\mathbb{C}=\mathbb{Q}_\omega$. (About string amplitudes see book Green, Schwartz and Witten [12] and references therein.)

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In spite of the beauty of formulae (1.2.) and (7.2.) infinite products in them diverge for all admissible arguments (s, t, u).

It provides the following three problems.

1. *How to regularize the adelic product?*
2. *How to extend them to others algebraic number fields?*
3. *How to extend them to ramified quasicharacters?*

We give a short survey of solutions of the posed problems based on recent activity in p-adic analysis (see books by Schikhof [13], Weil [14], Gel'fand et al [15], Koblitz [16] and also [1,2]).

3. Regularization

For simplicity we consider a method of regularization to the simplest case of the field of rational numbers $\mathbb{Q}$, i.e. to the adelic product (1.2.) firstly exposed in Vladimirov [17] (1993). In this case the idele group $A^{\times}_{\mathbb{Q}}$ consists of elements

$$X = (x_\infty, x_2, x_3, \ldots, x_p, \ldots), x_p \in \mathbb{Q}_p^{\times}, p = \infty, 2, 3, \ldots,$$

$$x_p \in Z_p^{\times},\ p \leq N, \tag{1.3.}$$

where $Z_p^{\times}$ is the group of unities of field $\mathbb{Q}_p$; number N depends only on idele X.

Product (1.2.) refers in fact to the beta-functions (2.2.) and (3.2.) of fields $\mathbb{Q}_p$ which in turn are products (4.2.) of three gamma-functions $\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma)$. Therefore we may restrict ourself to the case of gamma-functions $\Gamma_p(\alpha), \quad p = \infty, 2, 3, \ldots$. Now we note that

$$\Gamma_p(\alpha) = \int_{\mathbb{Q}_p} |x|_p^{\alpha-1}\chi_p(x)d_p x, \quad p = 2, 3, \ldots \tag{2.3.}$$

is the Mellin transform of test function

$$\varphi(x) = \tau(p - |x|)\chi_p(x) \in \mathcal{D}(\mathbb{Q}_p)$$

for which the Mellin transform of its Fourier transform $\tilde{\varphi}(\xi)$ is equal to 1 (see [17]). This fact permits us to use Tate's formula (see, e.g., [14]), and we obtain the key formula for local gamma-functions of field $\mathbb{Q}$

$$\Gamma_\infty(\alpha)\prod_{p=2}^{P}\Gamma_p(\alpha)\prod_{p=P_1}^{\infty}(1 - p^{-\alpha})^{-1}$$

$$= \prod_{p=P_1}^{\infty}(1 - p^{\alpha-1})^{-1}, \quad \alpha \neq 1 + 2\pi i/\ln p Z, \quad P = \infty, 2, 3, \ldots. \tag{3.3.}$$

(Here P_1 is next prime number after P.)

If we denote (if exists)

$$\mathrm{reg}\prod_{p=2}^{\infty}\Gamma_p(\alpha) = \lim_{P\to\infty}\prod_{p=2}^{P}\Gamma_p(\alpha)\prod_{p=P_1}^{\infty}(1 - p^{-\alpha})^{-1},$$

so from (3.3.) it follows the regularized adelic formulae for gamma-functions

$$\Gamma_\infty(\alpha)\,\mathrm{reg}\prod_{p=2}^{\infty}\Gamma_p(\alpha) = 1. \tag{4.3.}$$

It exists for $\operatorname{Re}\alpha < 0$ and it is defined by analytic continuation for $\operatorname{Re}\alpha \geq 0$ (in spirit of ideas of Hadamard, M.Riesz, Bogolubov, Gel'fand, ...).

If we denote

$$\operatorname{reg}\prod_{p=2}^{\infty} B_p(\alpha,\beta,\gamma) = \prod_{x=\alpha,\beta,\gamma} \operatorname{reg}\prod_{p=2}^{\infty}\Gamma_p(x) \tag{5.3.}$$

and use the regularized adelic formula (4.3.), we can represent the formal equality (1.2.) in the following regularized adelic form (see Vladimirov [17])

$$V(s,t,u)\operatorname{reg}\prod_{p=2}^{\infty} V_p(s,t,u) = 1, \quad s+t+u=-8, \tag{6.3.}$$

Thus, a regularization exists (Vladimirov [17]).

4. Extension to fields of algebraic numbers

Let $\mathbb{Q}(\varepsilon)$ be a field of algebraic numbers generated by algebraic number ε of degree n and let the minimal polynom for it has σ and 2τ real and complex roots respectively, so $n = \sigma + 2\tau$. Then similar to (4.3.) the key adelic regularized formula for gamma-functions of the field $\mathbb{Q}(\varepsilon)$ follows

$$\Gamma_\infty^\sigma(\alpha)\Gamma_\omega^\tau(\alpha)\operatorname{reg}\prod_{p=2}^{\infty}\prod_{j=1}^{m_p}\Gamma_{q_{pj}}(\alpha) = |D|^{1/2-\alpha}, \tag{1.4.}$$

where D is discriminant of the field, m_p is a number of various prime divisors π_{pj} entering in decomposition of prime number p,

$$p = \pi_{p1}^{e_{p1}}\pi_{p2}^{e_{p2}}\dots\pi_{pm_p}^{e_{pm_p}}, \quad p = 2,3,\dots, \tag{2.4.}$$

$q_{pj} = p^{f_{pj}}$ is the module of a local p-field which correspond to the place pj. Positive integer e_{pj} and f_{pj} satisfy the equality

$$e_{p1}f_{p1} + e_{p2}f_{p2} + \dots + e_{pm_p}f_{pm_p} = n. \tag{3.4.}$$

(On algebraic number theory see Borevich and Shafarevich [3], Weil [14], Lang [21],...).

From the key formula (1.4.) it follows regularized adelic formula for the Veneziano and Virasoro amplitudes (see Vladimirov [18,19] and Vladimirov and Sapuzhak [20])

$$V^\sigma(s,t,u)W^\tau(4s,4t,4u)\operatorname{reg}\prod_{p=2}^{\infty}\prod_{j=1}^{m_p} V_{q_{pj}}(s,t,u) = \sqrt{|D|},$$
$$s+t+u=-8. \tag{4.4.}$$

Special explicit forms of adelic formulae (4.4.) has been derived for quadratic fields $\mathbb{Q}(\sqrt{d})$ [18]; m-circular fields $\mathbb{Q}(\exp 2\pi ik/m)$, $(k,m)=1$ [20,19]; and for some cubic fields $\mathbb{Q}(\sqrt[3]{d})$, d-square-free $d \not\equiv \pm 1$ [19].

5. Extension to ramified quasicharacters

In view of complexity of the problem we restrict ourself by considering the simplest case, namely group $A_{\mathbb{Q}}^\times$ of ideles $X = $ (1.3.) of field $\mathbb{Q}$. The general form of a quasicharacter of group $A_{\mathbb{Q}}^\times$ which is trivial on principal ideles (isomorphic to $\mathbb{Q}^\times$) is [15]

$$\Theta(X;\alpha) = \operatorname{sgn}^\nu x_\infty \prod_{p=2}^{\infty}\theta_p(x_p)|x_p|_p^{i\alpha_p}|X|^\alpha, \nu\in F_2, \alpha\in\mathbb{C}, \tag{1.5.}$$

($\theta_p(p) = 1$ and α_p are real), if and only if, the following conditions are fulfilled:

$$\theta(-1) = 1, \quad \theta(p) = p^{i\alpha_p}, p = 2, 3, \dots \tag{2.5.}$$

where θ is the character of the group $A^{\times}_{\mathbb{Q}}/\mathbb{Q}^{\times}$ of the form

$$\theta(X) = \mathrm{sgn}^{\nu} x_{\infty} \prod_{p=2}^{\infty} \theta_p(x_p). \tag{3.5.}$$

(F_2 is field of integers on mod 2.)

Using this result, we may derive adelic formula for beta-functions

$$B_{\infty}(\alpha, \nu; \beta, \mu; \gamma, \eta), \quad \alpha + \beta + \gamma = 1, \nu + \mu + \eta = 0, \nu, \mu, \eta \in F_2. \tag{4.5.}$$

Beta-function $B_{\infty}(\alpha, \dots, \eta)$ is constructed by a standard way for quasicharacters $\Theta(X; \alpha)$ and $\Pi(X; \beta)$ of the form (1.5.) with local parameters: μ, π_p, β_p. By the formula (3.5.) we construct the character π and than the character σ such that $\theta\pi\sigma = 1$. At last we suppose that ranks of local characters $\theta_p, \pi_p, \sigma_p$ are equal, i.e.,

$$\rho_p = \rho(\theta_p) = \rho(\pi_p) = \rho(\sigma_p), \quad p = 2, 3, \dots. \tag{5.5.}$$

So the norms of the conducter ideal of characters θ, π, σ are equal and we denote them by

$$N = \prod_{p=2}^{\infty} p^{\rho_p}. \tag{6.5.}$$

Under this conditions the adelic formula for beta-function exists and has the following explicit form (Vladimirov [25])

$$\Delta\sqrt{N} = B_{\infty}(\alpha, \nu; \beta, \mu; \gamma, \eta)$$
$$\times \operatorname{reg} \prod_{p=2}^{\infty} \frac{1 - p^{\alpha-1}\theta(p)}{1 - p^{-\alpha}\bar{\theta}(p)} \frac{1 - p^{\beta-1}\pi(p)}{1 - p^{-\beta}\bar{\pi}(p)} \frac{1 - p^{\gamma-1}\sigma(p)}{1 - p^{-\gamma}\bar{\sigma}(p)}, \tag{7.5.}$$

where Δ is some complex number not depending on α, β, γ, $|\Delta| = 1$.

Similar results has been obtained for one-class quadratic fields $\mathbb{Q}(\sqrt{d})$ (Vladimirov [25]). For $d < 0$ there is only 9 such fields, namely

$$d = -1, -2, -3, -7, -11, -19, -43, -67, -163; \tag{8.5.}$$

for $d > 0$ there exists infinite numbers such fields, e.g., $d = 2, 3, 5, 6, 7, \dots$ (see [3]). To describe quasicharacters on idele group which is trivial on the group of principal ideles $\mathbb{Q}^{\times}(\sqrt{d})$ it was necessary to study the structure of prime divisors of one-class field $\mathbb{Q}(\sqrt{d})$. It was made in Vladimirov [24]. The corresponding adelic formulae have rather complicate form, and we refer the reader to the original paper (Vladimirov [25]).

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