The Universality of Confining Potential and the Running of the Quasi-Coulombic Potential Constant in the Independent-Quark Model

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Parameters of the QCD-motivated static potential and the quark masses are calculated on the basis of the 1^{--} meson mass spectra in the framework of the relativistic independent-quark model based on the Dirac equation. The value of the confining potential parameter is found to be (0.20 ± 0.01) GeV^2 for interactions between quarks and antiquarks independently on their flavors. The flavor independence of the confining potential is justified on the $5 \cdot 10^{-2}$ accuracy level both for the heavy quarks and for the light ones. The values of parameter α_s which is a strength of the quasi-Coulombic potential are consistent with the QCD-motivated decrease of α_s at small interaction range. The $q\bar{q}$ separation inside light and heavy vector mesons V and the ratios of $V \to e^-e^+$ decay widths for heavy vector mesons are evaluated.

1. Introduction

The calculation of masses and decay widths of various hadronic states still remains among the unsolved problems of quantum chromodynamics (QCD). Even if we restrict ourselves only to consideration of $q\bar{q}$ bound states, ignoring baryons and exotic multiquark states, we, nevertheless, run into some technical and conceptual difficulties such as high order perturbative calculations and the confinement phenomenon. Moreover, it seems probable that within any constructive formulation of the bound-state problem in QCD, only the simplest cases will be of a rigorous analysis. For instance, although recently a considerable progress in lattice QCD was achieved (see, e.g. Ref. [1]), the calculations of mass values for all hadrons (including the light ones) on the basis of the first principles of QCD are not possible with precision compatible with existing data [2] and demand too much computation time.

Mainly for these reasons a set of phenomenological hadron models is used now which differ from each other both in basic assumptions and in a precision of calculations. Since the phenomenological models involve adjustable assumptions they do not provide a controlled approximation scheme for QCD but it seems that the model calculations are helpful for understanding the unsolved theoretical problems (such as the confinement or the bound state problem) as well as for the interpretation of data. For instance, the mass spectrum of the J/Ψ quarkonia is known to be described for the first time in the framework of the nonrelativistic potential quark model. This model is the simplest one among the phenomenological models of hadrons and is suitable for the heavy quark hadrons. However the consideration of mesons containing the light quarks is more complicated task, and it demands the relativistic as well as nonpotential effects to be taken into account (see, e.g. Refs. [3-10]).

In the present paper we deal with a description of some characteristics of both the light and heavy vector mesons in the framework of a phenomenological relativistic model with independent quarks. It is well known that the principle of the independent motion of each constituent in the mean steady-state field is the main statement of models of the considered type. We use the Dirac equation with a static QCD-motivated potential to describe the quark or antiquark motion in the mean field inside the meson. Our model does not contradict first principles of QCD and allows one in a simple fashion to carry out numerical calculations of the meson characteristics, such as masses and separations between quarks and antiquarks. Moreover, it is known, that the independent quark model is suitable for hadrons with any number of both fermion and boson constituents. It seems that as the parton model is an adequate approximation for the perturbative QCD in hard processes, so the independent-quark model is relevant to the QCD bound state problems in the analogous manner. One of the translation invariant versions of this model was presented in Ref. [9], where it was suggested on the basis of the data for hadron mass spectra that the motion of each valent constituent obeyed phenomenological selection rules with respect to their radial quantum numbers and angular moments. In the framework of this model the generalizations of the Veneziano-Nambu and Chew-Frautchi formulae for light mesons were obtained, as well as the flavor independence of the confining potential was confirmed [8-10]. However, the value of the strong coupling constant α_s was 'frozen' at small interaction distances for all quark flavors and was equal approximately to 0.32. This was caused by the fact that actually in Refs. [8-10] a phenomenological radial equation was used with a simple potential which took into account the scalar confining and vector quasi-Coulombic interactions. In the present paper we go on from the phenomenological equation to the Dirac equation with a sum of the scalar linearly rising potential and the vector quasi-Coulombic one, and we get a variation of α_s which depends on the quark flavor. This variation is consistent qualitatively with the QCD asymptotic freedom phenomenon. Moreover, the coefficient for the confining part of the potential remains the same for all quark flavors and its value is $\sigma = (0.20 \pm 0.01)$ GeV^2. This result supports the hypothesis of the universality of confinement for all color objects.

The article is organized as follows. We briefly review the basic equations and the main statements of the considered model in Section 2. Here the Dirac equation in the external field, consisting of the Lorentz scalar and vector components, is transformed into a form suitable for further numerical calculations. The method of the numerical calculations as well as the calculated parameters of $q\bar{q}$ interaction, masses, $q\bar{q}$ separations inside vector mesons and the ratios of $V \to e^- e^+$ decay widths for heavy vector mesons V are presented in Section 3. We discuss the obtained results in the last Section 4.

2. Basic model statements

According to the main statement of the independent-quark model the hadron is considered as a system composed of a few, let say N, non-interacting with each other directly, valent constituents (quarks and antiquarks) having the coordinates $\mathbf{r}_i$, $i = 1, ..., N$, and moving in some mean field. One supposes that this field is a white confining field which is produced by the constituents and takes into account the effects of creation and annihilation of a sea of $q\bar{q}$ pairs as well. Furthermore, to simplify calculations it is assumed that this external mean field is spherically symmetric and its motion in space is determined by motion of its center with the coordinate $\mathbf{r}_0$. In other words one treats the mean field as a quasi-classical object possessing some energy ϵ_0. Meantime each of N constituents interacting with the spherical external field gets the state with a definite value of its energy ϵ_i, so that the hadron mass can be evaluated as

$$M_h = \epsilon_0 + \epsilon_1 + ... + \epsilon_N. \tag{1}$$

In the equal-time approach the entire system, the hadron, is described by a stationary wave function which in the so-called translation-invariant model can be represented as follows:

$$\Psi_{M_h}(\mathbf{r}_0, \mathbf{r}_1, ..., \mathbf{r}_N) \propto \psi_1(\mathbf{r}_1 - \mathbf{r}_0)...\psi_N(\mathbf{r}_N - \mathbf{r}_0). \tag{2}$$

In order to keep out the ghost-motion states of the independent constituents center of inertia, one usually associates the coordinate of the constituents center of mass with the coordinate $\mathbf{r}_0$ of the mean field center. Thus, in this model any constituent turns out to be coupled to the others via the common mean field which is moving together with them. The wave function $\psi_i(\mathbf{r}_i - \mathbf{r}_0)$ for any constituent is a solution of the single-particle equation with the mean field static potential $U(\mathbf{r}_i - \mathbf{r}_0)$ which is chosen to be spherically symmetric. Angular dependence of the single-particle wave functions in a stationary state may be separated in a well-known manner, and in order to evaluate the spectroscopical hadron characteristics it is necessary to solve the radial equations with the model potentials for each constituent. Note that it is impossible to evaluate ϵ_0 without additional assumptions, and this quantity is a phenomenological parameter in the considered model.

Hereafter we shall consider mesons as two-body hadrons consisting of a quark and an antiquark. More precisely, the meson is considered as a bound system of the quark (antiquark) q_1 and the antiquark (quark) q_2 in the $n^{2S+1}L_J$ state. On the phenomenological ground the mass formula (1) for such the system can be presented in the following form [9]:

$$M(n, J^{PC}) = E_1(n_1^r, j_1) + E_2(n_2^r, j_2), \tag{3}$$

where $E_i(n_i^r, j_i)$, $i = 1, 2$, is called the energy spectral function or the mass term for the quark (antiquark) q_i and will be defined below. Note that each mass term contains a part of the mean field energy ϵ_0. The meson parity $P = (-1)^{L+1}$ and the eigenvalue C of charge conjugation for the neutral meson of $q\bar{q}$ type is equal to $(-1)^{L+S}$ where the total spin $S = 0$ or 1. In our notation m_i are the quark (antiquark) masses and, for convenience, we choose the ordering when $m_1 < m_2$, n_i^r and j_i are the radial quantum numbers and the quantum numbers of the effective angular moments.

The main ansatz for evaluating the meson masses in the framework of our model is the following phenomenological expression for the energy spectral function:

$$E_i(n_i^r, j_i) = [\lambda_i + {m_i}^2]^{1/2} + c[1 + (-1)^{L+j_i+1/2}]. \tag{4}$$

The first term represents the relativistic effective energy of the constituent moving in the mean field inside the meson, m_i being a model parameter and λ_i being found as an eigenvalue of the radial relativistic equation of the model. The second term in the formula

(4) includes a part of energy of the mean field and is purely phenomenological with a model parameter $c = 0.035$ GeV. It gives small corrections to the spectra of different meson families. When one compares the evaluated meson masses with data it can be found that the best agreement is achieved under condition that the quantum numbers n_i^r and j_i obey the following selection rules [9]:

$$\begin{aligned} j_1 &= j_2 + 1 = J + 1/2, J \neq L + S, \\ j_1 &= j_2 = J - 1/2, J = L + S, \\ n_1^r &= n_2^r = n - 1. \end{aligned} \tag{5}$$

Within the scope of this model the wave function for any constituent (the quark/antiquark orbital) is a solution of the single-particle equation with a static potential. We choose the Dirac equation with the QCD-motivated potential $V(r)$ for description of the interaction of the quark or antiquark with the external mean field in order to determine the orbitals inside the meson. This potential is spherically symmetrical and consists of the Lorentz scalar and vector parts: $V(r) = V_0(r) + \beta V_1(r)$. Hence the equation for a fermionic constituent has the form

$$E_i \psi_i(\mathbf{r}) = \left[(\boldsymbol{\alpha}\mathbf{p}) + \beta(m_i + V_0) + V_1 \right] \psi_i(\mathbf{r}) \tag{6}$$

with $V_0(r) = \sigma r/2$ and $V_1(r) = -2\alpha_s/3r$, where the two model parameters σ and α_s are introduced and have got meanings of the string tension and the strong coupling constant at small distances, correspondingly. It is well-known that the solutions of Eq. (6) with the total angular momentum j and its projection m can be represented as

$$\psi_i(\mathbf{r}) \propto \begin{pmatrix} f_i(r)\Omega_{jl}^m(\theta, \varphi) \\ -ig_i(r)(\boldsymbol{\sigma}\mathbf{n})\Omega_{jl}^m(\theta, \varphi) \end{pmatrix}, \tag{7}$$

where $\mathbf{n} = \mathbf{r}/r$. If $k = -\omega(j + 1/2)$ where ω is an eigenvalue of the space-parity operator, the system of the radial Dirac equations for the i-fermion in the mean field with the definite energy sign and spin projection reads

$$\begin{aligned} (E_i - V_0 - V_1 - m_i) f_i &= -\frac{k+1}{r} g_i - g_i', \\ (E_i + V_0 - V_1 + m_i) g_i &= -\frac{k-1}{r} f_i + f_i'. \end{aligned} \tag{8}$$

Using Eqs. (8) one can derive the second order equation for the 'large' component $f_i(r)$, and then making a substitution

$$\varphi_i(r) = r f_i(r) \left[V_0(r) - V_1(r) + m_i + E_i \right]^{-1/2} \tag{9}$$

one comes on to the equation for $\varphi_i(r)$ in the following form:

$$\varphi_i'' + \lambda_i \varphi_i = \Bigg[(m_i + V_0)^2 - (E_i - V_1)^2 + \frac{k(k-1)}{r^2} + \frac{3(V_0' - V_1')^2}{4(E_i - V_1 + V_0 + m_i)^2} -$$

$$- \frac{k(V_0' - V_1')}{r(E_i - V_1 + V_0 + m_i)} - \frac{(V_0'' - V_1'')}{2(E_i - V_1 + V_0 + m_i)} \Big] \varphi_i. \quad (10)$$

Further on we restrict ourselves evaluating only characteristics of the radial excitations of the S-wave 1^{--} mesons because they are described by the simplest version of model and supported by the most extensive set of accurate data [2], especially for heavy mesons. So, $k = 1$ and λ_i entering Eq.(4) can be calculated with the help of the S-wave radial equation:

$$\varphi_i'' + \lambda_i \varphi_i = \Big\{ - \frac{4\alpha_s\sqrt{\lambda_i + m_i^2}}{3r} - \left(\frac{2\alpha_s}{3r}\right)^2 + m_i\sigma r +$$
$$+ \left(\frac{1}{2}\sigma r\right)^2 - \left[\frac{1}{2}\sigma r \left(m_i + \sqrt{\lambda_i + m_i^2}\right) + \left(\frac{1}{4}\sigma r\right)^2 + \frac{5}{6}\alpha_s\sigma - \frac{\alpha_s^2}{3r^2}\right] \Big\}$$
$$\left[\frac{2}{3}\alpha_s + r\left(m_i + \sqrt{\lambda_i + m_i^2}\right) + \frac{1}{2}\sigma r^2\right]^2 \Big\} \varphi_i. \quad (11)$$

The right-hand side of Eq. (11) has a singularity at the origin, and when $r \to 0$ it behaves as $3/4r^2 - 4\alpha_s^2/9r^2$. Therefore one should keep $\alpha_s < 3/2$ in order to prevent a fall at the origin. At the infinity $r \to \infty$ the effective potential behaves as the oscillatory one and tends to $\sigma^2 r^2/4$.

In the framework of our model the quantities m_i, σ and α_s are phenomenological and to be determined as a result of calculations of meson spectra from Eq. (11) and a comparison of them with experimental data.

3. Numerical evaluation of the strong interaction parameters and the 1^{--} meson characteristics

The model equation (11) presented in the preceding section can be solved only by numerical methods. In this section we present results on the numerical evaluation of m_i, σ and α_s parameters , as well as some 1^{--} meson characteristics. For calculating the eigenvalues of Eq. (11) a computer code has been worked out. The code algorithm is based on the Numerov three point recurrent relation [11]:

$$y_{i-1} = \left[y_i\left(2 + \frac{5}{6}F(r_i, E)h^2\right) - y_{i+1}\left(1 - \frac{F(r_{i+1}, E)h^2}{12}\right)\right] \times$$
$$\times \left[1 - \frac{F(r_{i-1}, E)h^2}{12}\right]^{-1}, \quad (12)$$

where $y'' = F(r, E)y$. For bound states one can put $y(r)$ equal to zero for the values of $r \geq r_{max}$. We put $r_{max} \approx 10r_{cl}$ where r_{cl} is a classical radius of the bound state which is determined from the equation $U(r_{cl}) = E$ and E is the initial energy of the considered level. Then the value of E can be determined from the condition: $y(0) = 0$. However the $F(r, E)$ has a singularity at the origin. So when we calculate $y(0)$ with the help of formula (12) we use a regularization procedure which is analogous to the procedure which was worked out, for instance, in Ref. [12].

In order to estimate the precision of the calculation algorithm the well-known test with diminished spacing h is used. The obtained results allow us to disregard machine calculation errors as compared with systematic errors of the model. The accuracy criterion for the fitted parameters was a limiting value of acceptable errors for evaluated hadron masses as compared with typical experimental errors which we choose to be less than or equal to 50 MeV. Thus, the parameter errors written below must be considered as our estimation of systematic errors of the model.

When calculating the values of the model parameters we do not suppose *a priori* the validity of the hypothesis of flavor independence for the confining potential. First of all the model parameters m_i, α_s and σ for the $b\bar{b}$ and $c\bar{c}$ radial excitation of 1^{--} states were found. The fit was carried out for each meson family independently. The existing experimental data allowed to find these parameters as well as to prove the validity of the model. In this manner it was found that the value of σ is the same for the $b\bar{b}$ and $c\bar{c}$ states within the systematic errors of the model and equal to (0.20 ± 0.01) GeV^2. Taking into account that there are no well established data for the higher radial excitations of 1^{--} light mesons, the obtained σ value was used for calculating m_i and α_s for light quarks. Note that when doing the calculations we neglect the isotopic mass splitting of the u- and d-quarks because it is beyond of the model accuracy.

With use of this procedure the following results were obtained. The coupling α_s (quasi-Coulombic potential constant) as well as the quark masses were evaluated for each family of the 1^{--} mesons which are composed of the quark and antiquark with u-, d-, s-, c- or b-flavors. Within the model accuracy all results for the well known experimental data on the 1^{--} mesons are consistent with the following parameter values:

$$\begin{aligned} m_{u,d} &= \left(0.008^{+0.01}_{-0.007}\right) \text{ GeV}, & \alpha_{s,ud} &= 0.6 \pm 0.25, \\ m_s &= (0.16 \pm 0.04) \text{ GeV}, & \alpha_{s,s} &= 0.53 \pm 0.15, \\ m_c &= (1.34 \pm 0.05) \text{ GeV}, & \alpha_{s,c} &= 0.31 \pm 0.03, \\ m_b &= (4.65 \pm 0.05) \text{ GeV}, & \alpha_{s,b} &= 0.23 \pm 0.02. \end{aligned}$$

The input restriction of this model for quark mass values includes the inequality: $m_q > 0$, so the lower value of systematic error for $m_{u,d}$ is determined by this restriction. The calculated vector meson masses are shown in Table 1 in comparison with the experimental values. We also show the values for the $q\bar{q}$ separations inside vector mesons R_n (n is a number of the radial state) which have been numerically evaluated with the help of the following formula:

$$R_n^2 = \int \Bigg\{ (E_n + m + \sigma r/2 + 2\alpha_s/3r)\varphi_n^2(r) +$$

$$+ (E_n + m + \sigma r/2 + 2\alpha_s/3r)^{-1} \left[\varphi_n' + \left(\frac{\sigma/4 - \alpha_s/3r^2}{E_n + m + \sigma r/2 + 2\alpha_s/3r} - \frac{1}{r} \right) \varphi_n \right]^2 \Bigg\} r^2 dr. \quad (13)$$

Moreover, we have managed to estimate the $V \to e^+e^-$ decay widths of the vector mesons using the well-known nonrelativistic formula [3-5]:

$$\Gamma(V \to e^+e^-) = 16\pi\alpha^2 e_q^2 \frac{|\Psi(0)|^2}{M_V^2} \left(1 - \frac{16\alpha_s}{3\pi}\right)^2. \quad (14)$$

In our case the square of the wave function at the origin in nonrelativistic approximation, as it is seen from relation (2), is proportional to the product of the squared large components $f_{in}(0)$. Thus, taking into account normalization factors of the meson wave function we get

$$\Gamma(V_n \to e^+e^-)/\Gamma(V_m \to e^+e^-) = \frac{M^5_{Vm}}{M^5_{Vn}} \left(\frac{f_{n1}(0) f_{n2}(0)}{f_{m1}(0) f_{m1}(0)} \right)^2 . \tag{15}$$

The results obtained for R_n and $\Gamma(V_n \to e^+e^-)/\Gamma(V_m \to e^+e^-)$ are presented in Tables 1 and 2. In some cases one can see a disagreement between the calculated decay ratios and the experimental values which might be caused by the fact that the values of $f_{ni}(r)$ are very sensitive to the behavior of the model potential at $r \sim 0$, and a small variation of α_s brings to the considerable variation of decay widths. Note that the constants α_s are set the same for different radial states belonging to some meson family. We suppose to clarify this problem in our future papers.

Table 1. Evaluated masses of the 1^{--} mesons in comparison with the data from Ref. [2] and the $q\bar{q}$ separations inside vector mesons in comparison with the results of Ref. [1].

Meson	M_{exp} [MeV]	M_{th} [MeV]	R_n^{th}[fm]	$R_n^{[1]}$
ρ	769.9±0.8	760	0.68	-
ρ'	1465±25	1455	1.06	-
ϕ	1019.413±0.008	1010	0.61	-
ϕ'	1680±50	1690	1.0	-
J/ψ	3096.88±0.04	3100	0.41	0.43
ψ'	3686.00±0.09	3670	0.76	0.85
ψ''	4040±10	4070	1.04	1.18
ψ'''	4415±6	4410	1.26	1.47
Υ	9460.37±0.21	9505	0.25	0.24
Υ'	10023.3±0.3	9980	0.51	0.51
Υ''	10355.3±0.5	10310	0.72	0.73
Υ'''	10580±3.5	10590	0.90	0.93

Table 2. Evaluated ratios of the the $V \to e^+e^-$ decay widths in comparison with the data from Ref. [2].

V_i/V_j	$(\Gamma_i/\Gamma_j)^{min}_{exp}$	$(\Gamma_i/\Gamma_j)^{max}_{exp}$	$(\Gamma_i/\Gamma_j)^{min}_{th}$	$(\Gamma_i/\Gamma_j)^{max}_{th}$
ψ/ψ'	2.08	2.92	2.21	2.98
ψ/ψ''	5.43	9.38	3.72	4.94
ψ/ψ'''	8.58	15.22	4.96	6.54
ψ'/ψ''	2.14	3.92	1.46	1.89
ψ'/ψ'''	3.39	6.35	1.96	2.51
ψ''/ψ'''	1.05	2.43	1.18	1.49
Υ/Υ'	2.30	2.81	2.92	3.23
Υ/Υ'''	4.55	6.31	6.05	6.69
Υ'/Υ'''	1.75	2.54	1.97	2.18

4. Conclusions and discussion

In this paper we consider the relativistic hadron model with independent quarks and evaluate parameters of the $q\bar{q}$ model potential, as well as the masses and the $q\bar{q}$ separation inside the 1^{--} mesons. We find that the value of $\sigma = (0.20 \pm 0.01)$ GeV^2 is the same for u-, d-, s-, c- and b-quark flavors within the systematical errors of the model. It is the main result of our paper which confirms the flavor independence of confining potential on the $5 \cdot 10^{-2}$ level of accuracy. If one uses as definitions of the characteristic confinement scales the confinement length l_c and mass m_c so that [13]: $\sigma = m_c/l_c$, $m_c l_c = 1$, then one will obtain: $m_c = (0.447 \pm 0.011)$ GeV, $l_c = (0.442 \pm 0.011)$ fm. In addition, the correspondence to the QCD asymptotic freedom phenomenon, or the decrease of α_s at shorter distances, is confirmed both for the light and heavy quarks in this model. Thus, in the framework of the independent-quark model on the basis of existing data for the 1^{--} meson mass spectra we have shown the universality of the confining potential of interaction between quarks and antiquarks and at the same time the diminishing of α_s with a quark mass growth. Within systematical errors of this model it is difficult to prove the logarithmic dependence of α_s on Q^2, moreover, there are considerable nonperturbative contributions in the ultraviolet region (see, e.g. Refs. [14,15])

The method presented allows one to calculate the spectroscopical meson characteristics such as the $q\bar{q}$ separations inside the light and heavy vector mesons and the ratios of the $V \to e^-e^+$ decay widths in a simple fashion. For instance, the values of $q\bar{q}$ separations for the heavy vector mesons are in agreement with those obtained in Ref. [1], where the calculations were performed on the basis of lattice QCD, while the calculations of the light meson characteristics lie now beyond possibilities of the lattice QCD approach. The model parameters obtained in this work, such as m_i, σ and α_s, may be compared with the values obtained by other methods and on the ground of the QCD first principles (see, e.g. Refs. [1-7]). One can see that our parameters do not contradict to these values and lie in the generally used range. Besides, the advantage of the presented method consists in its applicability to composite systems with any number of constituents, such as baryons and exotic multiquark mesons.

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