The Superposition Principle and Conservation Laws in Quantum Mechanics

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1. Introduction

The conservation laws are connected very closely with transformation properties of the observables under coordinate system transformations. The circumstance that coordinate transformations mapping the space-time properties form groups appears in the observables - they have some group-theoretic properties too. The Noether theorems establish one-to-one correspondence [1] between these properties of observables and conservation laws therefore their fulfilment is the mapping of space-time properties.

One of the main important object of quantum mechanics is the wave function. It can not be observed in the experiment but the wave function and their derivatives form the bilinear Hermitian forms defining the observables therefore the wave function and their derivatives have some group-theoretic properties connected with conservation laws too. If some quantum mechanics scheme is formulated in terms of amplitudes or propagators then these objects also are obliged to have the correspondent group-theoretic properties too due to fulfilment of conservation laws. It is needed to take into account that the quantum mechanics is the local gauge theory [2] therefore the transformation of unobserved objects accept more wide transformations then it is accepted for observed variables — the last ones have to be unique.

Thus, the fulfilment of conservation laws in quantum theory is the consequence (from mathematical point of view) of group-theoretic requirements to the wave functions, amplitudes or propagators in accordance with the Noether theorems. It means from one hand that these objects have to be the sets forming groups, and from another hand the violation of group-theoretic requirements to the elemets of these sets must lead to the violation of the conservation laws.

At present three forms of quantum mechanics (Heisenberg, Schrodinger and Feynman forms) are considered to be equivalent therefore all conclusions obtained in some one form are valid in any another. The Feynman formulation of quantum mechanics is based on the expression [3]

$$\varphi_{ba} = \sum_c \varphi_{bc} \cdot \varphi_{ca} \tag{1.1}$$

for transition amplitudes φ_{fi} from initial state i into final one f.

The amplitudes φ_{fi} in the expression (1.1) are multiplicative (Markovian) for the successive path segments and they are additive for the alternative paths. Therefore two operations are used in the set of amplitudes (propagators): product due to multiplicativity and addition in accordance with the superposition principle*.

*In the accordance with [3,4] the term "superposition principle" is used here as the summation of alternative amplitudes.

The group in accordance with its definition is the monoid at the same time [5], i.e. the set with one operation. Therefore the expression (1.1) used as the base of the Feynman formulation (and two others which are equivalent to it) contains the rejection (explicitly or not) of consequently group-theoretic approach of quantum mechanics. In accordance with the Noether theorems it may lead to the violation of conservation laws.

The consecutive group-theoretic approach of quantum mechanics may be carried out only in the set with one operation — it is one of the necessary condition for that. At the same time as far as the Markovian properties of amplitudes (or propagators) are outside of doubts [6] then it is needed to extend the expression (1.1) so that a) both for successive paths and for alternative paths it should be used only one operation - multiplication, and b) the expression to be find out must turn to (1.1) under some conditions because of it is clear that this expression has the range of validity at least approximately.

The aim of this paper is the investigation of conservation laws fulfilment in the frameworks of two calculation schemes. One of them is based on the expression (1.1) which implies applicability of the ordinary (Euclidean) superposition principle and another is based on the consecutive group-theoretic approach leading to the non-Euclidean superposition principle on the Lobachevsky plane which may be considered as the generalization of the Euclidean one.

Both calculation schemes are applied to the "mental experiment" with two slits situated at the boundary of two media. This problem is the simplest one containing the noncommutativity of propagators along different paths due to the jump-like potential and requiring some rule of their composition due to two paths. In both schemes the observables are calculated and their transformation properties under coordinate system transformations as far as the fulfilment of conservation laws are investigated.

2. Observables

The aim to obtain the possibly most complete description of quantum mechanical phenomena leads to necessity of defining both the number of observable values and the number independent of them. We shall use for this purpose the circumstance that the Scrodinger equation

$$\nabla^2\chi(\mathbf{r}) + k^2(\mathbf{r})\cdot\chi(\mathbf{r}) = 0, \tag{2.1}$$

where $k^2(\mathbf{r}) = E - U(\mathbf{r}), U(\mathbf{r})$ is real potential, $\hbar^2 = 2m = 1$, is the second order differential equation in space variables over the set of complex function. We shall consider the stationary quantum mechanical problem as the Cauchy one defined at the initial point $\mathbf{r}_i$. The simplest and at the same time most general setting of the problem consists of such choice of conditions at the initial point which corresponds to an arbitrary scalar for $\chi(\mathbf{r}_i)$ and its arbitrary derivative along arbitrary direction $\mathbf{u}_i$

$$\chi(\mathbf{r}_i) = a\cdot\exp(ib),\ \mathbf{u}_i\cdot\nabla\chi(\mathbf{r}_i) = k_i\cdot c\ \exp(id), \tag{2.2}$$

where $\mathbf{u}_i$ is unit vector, $k_i = k(\mathbf{r}_i)$. Then it is needed to calculate the wave function and its derivative along arbitrary direction $\mathbf{u}_f$ at the final point $\mathbf{r}_f$.

The solution of such problem allows one to calculate the observables to be obtained at the final point for non-isotropic source situated at the initial point using integration over all directions there. In this way the vector observables at the final point may be either calculated along any arbitrary direction or averaged over all directions there.

The observables in quantum mechanics are presented by the bilinear Hermitian forms therefore as far as the initial conditions of the second order differential equation are defined by both the complex function and its first order derivative $\chi(\mathbf{r}_i)$ and $\nabla\cdot\chi(\mathbf{r}_i)$ correspondingly containing four real parameters a, b, c, d one has

$$\begin{aligned} j_0 &= k\chi\chi^* + (\mathbf{u}\nabla\chi)(\mathbf{u}\nabla\chi^*)/k, \quad j_1 = k\chi\chi^* - (\mathbf{u}\nabla\chi)(\mathbf{u}\nabla\chi^*)/k, \\ j_2 &= \chi(\mathbf{u}\nabla\chi^*) + \chi^*(\mathbf{u}\nabla\chi), \qquad j_3 = i(\chi(\mathbf{u}\nabla\chi^*) - \chi^*(\mathbf{u}\nabla\chi)), \end{aligned} \tag{2.3}$$

where all parameters, including $\mathbf{u}$, are given at $\mathbf{r}_i$. Four variables j_s in (2.3) have the current dimensionality, they satisfy with the identity

$$j_0^2 = j_1^2 + j_2^2 + j_3^2, \tag{2.4}$$

it is fulfill ed independently if $\chi(\mathbf{r})$ is the solution of (2.1) or not. Therefore, the identity (2.4) takes place at the initial and at the final points. It should be noted that the parameters in (2.3) are similar to the Stokes ones.

If we substitute (2.2) into (2.3) then one has

$$\begin{aligned} j_0 = (a^2 + c^2)k_i, \ j_1 = (a^2 - c^2)k_i, \\ j_2 = 2ac\cdot\cos(d-b)k_i, \ j_3 = 2ac\cdot\sin(d-b)k_i, \end{aligned} \tag{2.5}$$

(2.4) is fulfilled too. All j_s are real therefore they satisfy to the necessary requirements to observables.

Then it is clear that the observables in stationary problem of quantum mechanics described with equation (2.1) consist of four real values $j_s (s = 0, 1, 2, 3)$

$$\begin{aligned} j_0 = k\chi\chi^* + (\nabla\chi)(\nabla\chi^*)/k, \quad j_1 = k\chi\chi^* - (\nabla\chi)(\nabla\chi^*)/k, \\ \mathbf{j_2} = \chi(\nabla\chi^*) + \chi^*(\nabla\chi), \qquad \mathbf{j_3} = i(\chi(\nabla\chi^*) - \chi^*(\nabla\chi)), \end{aligned} \tag{2.6}$$

two of them are scalars (with $s = 0, 1$) and two of them are vectors (with $s = 2, 3$). They satisfies with the condition (2.4) too therefore only three of them are independent.

The transformations of vector observables $\mathbf{j}_2$ and $\mathbf{j}_3$ under the coordinate system rotation about angle θ around direction $\mathbf{n}$ belong to the group SO(3) [7,8].

Let us investigate the conservation laws of this problem. As far as they are connected with transformation properties of observables then the conservation laws investigation leads to the analysis of group-theoretic properties of the equation (2.1) solutions connected with observables by expressions (2.6).

With the aim of using the group-theoretic representations of the object transforming the union of χ and $\nabla\chi$ let us go over to their linear combinations $\Phi_+(\mathbf{r}), \Phi_-(\mathbf{r})$

$$\begin{aligned} \chi(\mathbf{r}) &= \tfrac{1}{\sqrt{2}}k^{-1/2}(\Phi_+(\mathbf{r}) + \Phi_-(\mathbf{r})), \\ \nabla\chi(\mathbf{r}) &= \tfrac{i}{\sqrt{2}}k^{1/2}(\Phi_+(\mathbf{r}) - \Phi_-(\mathbf{r}))\cdot\mathbf{u}(\mathbf{r}). \end{aligned} \tag{2.7}$$

Since there are introduced two complex functions Φ_+, Φ_- then the unit vector $\mathbf{u}(\mathbf{r})$ may be restricted with some requirements. We shall accept the following conditions

$$\nabla\mathbf{u}(\mathbf{r}) = 0, \ \mathbf{u}^2(\mathbf{r}) = 1. \tag{2.8}$$

Let us present the functions Φ_+, Φ_- as $\Phi(\mathbf{r}) = \text{column}\ \|\Phi_+(\mathbf{r}), \Phi_-(\mathbf{r})\|$ then the matrix Q transforming Φ at the initial point into Φ at the final point, $\Phi(\mathbf{r}_f) = Q(\mathbf{r}_f, \mathbf{r}_i) \cdot \Phi(\mathbf{r}_i)$, we shall call as "propagator".

One may derive the equation for Φ from (2.1), (2.7) and (2.8) ($\mathbf{k} = k\mathbf{u}$)

$$\nabla\Phi = [i\mathbf{k}\sigma_3 + \frac{\nabla k}{2k}\sigma_1]\Phi. \tag{2.9}$$

Let us consider four Hermitian, bilinear with respect to Φ_+, Φ_-, forms

$$j_s(\mathbf{r}) = \Phi^+(\mathbf{r}) \cdot \sigma_s \cdot \Phi(\mathbf{r}), \tag{2.10}$$

$s = 0, 1, 2, 3$, σ_0 is the unit matrix, other σ_s are Pauli matrices, Φ^+ is the Hermitian-conjugate to Φ. In accordance with (2.7) parameters j_0 and j_1 coincide with two first expressions and two second parameters j_2 и j_3 are the amplitudes in two last expressions in (2.3) or (2.6).

The transformation of vectors under coordinate system rotation described with matrices from the group $SO(3)$ is equal to the spinor transformation in the representation of the group $SU(2)$ with matrix $R = \exp[-i(\mathbf{n}\sigma)\theta/2]$ [8].

If one multiplies the identity

$$\exp[i(\mathbf{n}\sigma)\theta/2]\sigma_s \exp[-i(\mathbf{n}\sigma)\theta/2] = \cos\theta\sigma_s + (1-\cos\theta)n_s n_k \sigma_k + \sin\theta\varepsilon_{skp} n_k \sigma_p$$

from the left and right sides by Φ^+ and Φ respectively then the expression for the j'_s after rotation will have the form

$$j'_s = \cos\theta j_s + (1-\cos\theta)n_s n_k j_k + \sin\theta\varepsilon_{skp} n_k j_p, \tag{2.11}$$

where $s, k, p = 1, 2, 3$ and at the same time $\sum_{i=1}^{3} j_i'^2 = \sum_{i=1}^{3} j_i^2 = j_0^2$.

The rotation of coordinate system and the corresponding transformation of $\Phi = \text{column}\ ||\Phi_+, \Phi_-||$ leads to the same rotation of current vector as the acting of matrix from the group $SO(3)$ directly onto the current vectors $\mathbf{j}_2$ or $\mathbf{j}_3$. It is needed to mention that these vectors are collinear in the representation chosen here. The unit vector $\mathbf{u}$ defining the current direction is given by values j_1, j_2, j_3 in accordance with $\mathbf{u} = (j_1\mathbf{i} + j_2\mathbf{j} + j_3\mathbf{k})/j_0$. Thus, three bilinear Hermitian forms in (2.10) with $s = 1, 2, 3$ define the currents in three-dimensional space at the same time and the identity (2.4) is fulfilled due to the unitarity of R.

The transformation of the variable $\Phi = \text{column}\ ||\Phi_+, \Phi_-||$ under coordinate system rotations is described by the representation of the group $SU(2)$ and the observables are expressed as the bilinear Hermitian forms of its components, thus Φ is the spinor. As far as the variables j_s in (2.10) are completely similar to the four-vector components expressed in terms of spinor components [9,10] then they may be named as the four-current components.

One notes here that the space dimensionality of independent variables in equation (2.1) does not have the association to the number of current components j_s. For example, the expressions (2.3), (2.4) and (2.7), (2.10) have place in the one-dimensional case too (see also [11]).

Let us consider the conservation laws for observables which follow from the Schrodinger equation (2.1) or its spinor representation (2.9). Let construct the matrix $J = \Phi^+ \bigotimes \Phi$ obtained in accordance with the definition of matrix direct product. The matrix elements of J expressed through j_s from (2.10) are following

$$J = \Phi^+ \bigotimes \Phi = \|\Phi_+^*, \Phi_-^*\| \bigotimes \| \begin{matrix} \Phi_+ \\ \Phi_- \end{matrix} \| = \| \begin{matrix} \Phi_+^*\Phi_+ & \Phi_-^*\Phi_+ \\ \Phi_+^*\Phi_- & \Phi_-^*\Phi_- \end{matrix} \| =$$

$$= \frac{1}{2} \| \begin{matrix} j_0 + j_3 & j_1 - ij_2 \\ j_1 + ij_2 & j_0 - j_3 \end{matrix} \|. \tag{2.12}$$

Its determinant is equal to zero, J satisfies to the condition $J^2 = j_0 J$ which fits with the definition of idempotent matrix under the condition $j_0 = 1$, therefore the matrix J is similar to the density matrix of pure states.

If we differentiate (2.12) and use (2.9) together with Hermitian associated equation one has

$$\nabla J = ik\mathbf{u}\{\Phi^+ \bigotimes \sigma_3\Phi - \Phi^+\sigma_3 \bigotimes \Phi\} + \frac{\nabla k}{2k}\{\Phi^+ \bigotimes \sigma_1\Phi + \Phi^+\sigma_1 \bigotimes \Phi\}. \tag{2.13}$$

Going over to four-current j_s one derives the conservation laws in the differential form following from (2.9)

$$\sum_{s=0}^{3} \sigma_s \nabla j_s = 2k\mathbf{u}(\sigma_1 j_2 - \sigma_2 j_1) + \frac{\nabla k}{k}(\sigma_0 j_1 + \sigma_1 j_0). \tag{2.14}$$

The relations between four-current components accordingly to (2.14) may be expressed in the following form

$$\nabla j_0 = \frac{\nabla k}{k} j_1, \nabla j_1 = 2k\mathbf{u} j_2 + \frac{\nabla k}{k} j_0, \nabla j_2 = -2k\mathbf{u} j_1, \nabla j_3 = 0. \tag{2.15}$$

If we differentiate the identity (2.4) and use (2.15) one has the identity for relations (2.15) too. Therefore it allows one to consider (2.4) and (2.14) as the conditions of completeness for observables set $j_s, s = 0, 1, 2, 3$ for stationary Schrodinger equation with real potential.

It is clear from the definition of four-current j_s (2.3) and from the first and the second equations (2.15) that the linear combination of j_0 and j_1 forms the "probability density" $\rho = (j_0 + j_1)/2k$. The last one satisfies in accordance with (2.15) to the condition $\nabla \rho = j_2 \mathbf{u}$ ($\mathbf{j}_2 = j_2 \mathbf{u}$) which allows one to consider j_2 from (2.3) as the amplitude of "probability density" diffusion current. The current $\mathbf{j}_3$ from (2.6) to be conserved is collinear to the diffusion current in this representation, it may be considered as the convection current of "probability density".

Thus, three real variables ρ, $\mathbf{j}_2$ and $\mathbf{j}_3$ may be interpreted as "probability density", diffusion and convection currents respectively, they form the complete set of independent observables too in the problems described by stationary Schrodinger equation with real potential.

Let us define the group-theoretic properties of the complete propagator for spinor Φ under translations. They are defined by the properties of bilinear Hermitian forms (2.6), (2.10) and (2.15). The simplest way to obtain these properties is the analysis of the last equation from (2.15). The equation (2.1) leads to the current $\mathbf{j}_3$ from (2.6) conservation condition which has its differential form $\nabla \mathbf{j}_3 = 0$. The condition $\nabla j_3 = 0$ from (2.15) (or $j_3 = const$) is its expression for the spinor form of the Schrodinger equation (2.9) under condition (2.8).

Substituting the expression $\Phi(\mathbf{r}_f) = Q(\mathbf{r}_f, \mathbf{r}_i) \cdot \Phi(\mathbf{r}_i)$ into the conservation condition $j_3 = const$ one has

$$Q^{+}\sigma_3 Q = \sigma_3, \tag{2.16}$$

it defines the group-theoretic belonging of propagator Q under translations [8]: $Q \in SU(1,1)$ which satisfies to the conditions $detQ = 1, Q^*_{22} = Q_{11}, Q^*_{21} = Q_{12}$.

Thus the complete set of observables for equation (2.1) consists of four real parameters, only three of them are independent. They determine the density and two current vectors in three-dimensional space. Their transformation properties and conservation laws define the groups of transformation for spinor: the last one is transformed on the three-parameter group $SU(2)$ under the coordinate system rotations and on the three-parameter group $SU(1,1)$ under translations. The spinor Φ transformation by means of matrix σ_1 leads to the conservation of scalars j_0, j_1 and to change sign of vectors $\mathbf{j}_2, \mathbf{j}_3$ therefore σ_1 is the inversion operator [7].

3. Non-Euclidean composition of propagators

One of the significant peculiarity of quantum mechanics is the noncommutativity of operators (propagators) transforming a wave functions and their derivatives (or spinors). This peculiarity appears in the stationary problem to be considered in the fact that the complete propagator allowing to construct the complete spinor at the final point belongs to the non-Abelian group $SU(1,1)$. The metric and Gaussian curvature of the propagator logarithms space obtained in [7,11,12] show that this space is the Lobachevsky plane having the constant negative Gaussian curvature. The geometric image of the propagator is the geodesic vector on this plane at the same time.

The unidimensional Schrodinger equation was investigated in semiclassical region [13] by means of the product integral and this techiques allowed one to establish the association between the Schrodinger equation and the Lobachevsky geometry both in the semiclassical and anticlassical regions [14]. The belonging of the complete propagator of the equation (2.1) to the group $SU(1,1)$ (see section 2) coincides with the same of the unidimensional Schrodinger equation [14], it allows one to restrict with the last one to consider the role of the magnitude of the Gaussian curvature and its connection with (2.1). This opportunity is caused also by the fact that equation (2.1) is unidimensional one on any path between initial and final points with correspondent dependence of potential. At the same time the complete propagator may be constructed over the complete set of (partial) propagators calculated along each path in the accordance with the path integral concept [3].

Let us consider the unidimensional Schrodinger equation over the complex functions set with the conditions defined at the initial point x_0

$$\frac{d^2}{dx^2}\chi(x) + k^2(x) \cdot \chi(x) = 0 \ , \tag{3.1}$$

$k^2(x) = E - U(x),\ \hbar^2 = 2m = 1,$

$$\chi(x_0) = ae^{ib},\ \chi'(x_0) = k_0 ce^{id}\ . \tag{3.2}$$

$k(x_0) = k_0$. Let us go over from (3.1), (3.2) to two first order equations for linear combination of χ and χ' accordingly to

$$\Phi_{\pm}(x) = \frac{1}{\sqrt{2}} k^{1/2}(x) \cdot (\chi(x) \pm \frac{\chi'(x)}{ik(x)}), \tag{3.3}$$

compatible with (2.7).

One has, taking into account (3.1), two coupled equations for two-component value $\Phi = \|\Phi_+, \Phi_-\|$ joined in the matrix equation

$$\Phi'(x) = (ik\sigma_3 + \frac{k'}{2k}\sigma_1)\Phi(x) \tag{3.4}$$

with correspondent conditions at x_0, $\Phi^0 = \Phi(x_0)$ defined by (3.2). Its solution may be expressed ($dk = k'dx$) as [14]

$$\Phi(x) = Texp \int_{x_0}^{x} (ikdx\sigma_3 + \frac{dk}{2k}\sigma_1)\Phi^0 = Q(x, x_0)\Phi(x_0). \tag{3.5}$$

Let us consider the properties of propagator Q in (3.5). The matrix trace in the expression to be integrated is equal to zero then $detQ = 1$ everywhere. The Hermitian associated matrix has the form ($\sigma_s = \sigma_s^+$)

$$Q^+(x, x_0) = Texp \int_{x_0}^{x} (-ikdx\sigma_3 + \frac{dk}{2k}\sigma_1),$$

then one has

$$Q^+\sigma_3 Q = \sigma_3, \tag{3.6}$$

defining the group belonging of Q, $Q \in SU(1,1)$ [8]. Any matrix of this three-parameters group may be presented in the form

$$Q = e^{\vec{p}\vec{\sigma}} = e^{iM\sigma_3} e^{L\sigma_1} e^{iN\sigma_3} \tag{3.7}$$

with real parameters p_1, p_2, M, L, N and imaginary p_3. Let us consider $j_3(x)$ under condition $Q \in SU(1,1)$ taking into account (3.6) or (3.7)

$$j_3(x) \equiv \Phi^+(x)\sigma_3\Phi(x) = \Phi^+(x_0)Q^+\sigma_3 Q\Phi(x_0) = \Phi^+(x_0)\sigma_3\Phi(x_0) \equiv j_3(x_0),$$

that is $j_3(x) = const$. Therefore the conservation law for j_3 is the consequence of belonging Q to $SU(1,1)$.

Multiplying (3.4) by dx and going over to variables $u = 1/(2k), v = ix$ one has

$$d\Phi = dP \cdot \Phi = \frac{dv\sigma_3 - du\sigma_1}{2u} \cdot \Phi. \tag{3.8}$$

The matrix dP determinant taken with opposite sign defines the metric of propagator logarithms space in the coordinates (u, v)

$$det(dP) = -\frac{du^2 + dv^2}{4u^2} = -ds^2. \tag{3.9}$$

This expression defines the magnitude of the Gaussian curvature $C_G = -4$ of the Lobachevsky plane in its representation on the Poincare map.

It is well known that the spaces with the constant Gaussian curvature (positive, negative or zero) have the particular role in physical applications - only such spaces accept to obtain the necessary number of invariant values [15]. Taking this fact into account let us replace the integer 2 in (3.8) defining the Gaussian curvature of the Lobachevsky plane by some constant parameter R. If we return to the variables (k, x) one has the following equation instead of (3.4)

$$\frac{R}{2}\Phi' = (ik\sigma_3 + \frac{k'}{2k}\sigma_1) \cdot \Phi, \tag{3.10}$$

which goes to (3.4) under condition $R = 2$.

Now considering R as a constant let us go over to the second order equation in accordance with (3.3). It has the form

$$\chi'' + k^2(x)\chi + (\frac{2}{R} - 1)(\frac{k'}{2k} + ik)\chi' + (\frac{2}{R} - 1)(k^2 + \frac{i}{2}k')\chi = 0, \tag{3.11}$$

which turns under condition $R = 2$ to the Schrodinger one (3.1).

Thus the value of Gaussian curvature of the Lobachevsky plane defines the kind of differential equation. This equation, (3.11), corresponds to the Schrodinger one only in the case of $C_G = -4$, or $R = 2$ in (3.11). The deviation of Gaussian curvature from the value $C_G = -4$ leads to the equation (3.11) instead of initial Schrodinger one (3.1).

From the mathematical point of view the method of equation Schrodinger solving described above [7,11] consists of the going over from the partial differential equation (2.1) to the infinite set of ordinary differential equations along all paths between initial and final points. This approach corresponds to the algorithm of noncommutative integration of linear differential equations proposed in [17]. The conservation condition $j_3 = const$ is the analog of the Wronskian constancy for unidimensional equation (3.1). The method proposed here is near to the Feynman path integral with only the distinction that the propagators along each path belong to the noncommutative group $SU(1, 1)$. It means, from algebraic point of view, that these propagators are quaternions but not scalars and, from geometric point of view, that this propagators (more exactly - their logarithms) accept the geometric representation only in the spaces with nonzero Gaussian curvature taking into account their noncommutativity. This circumstance is the immediate consequence of the fact that the Schrodinger equation is the second order differential equation over the complex functions set.

The solution of problem (2.1) requires, besides of calculation of partial propagators along each path, some law of their composition into the complete propagator taking into account all paths. The requirements of conservation laws fulfilment and observables transformation lead to the complete propagator belonging to the group $SU(1, 1)$. As it was shown earlier [7,11,16] each partial propagator belongs to the same group therefore their composition allowing one to obtain the complete one has to conserve the group belonging. As far as alternative paths are equal then the complete propagator (for bosons) has to be independent of noncommutative partial propagators order in such their composition. The symmetric binary operation independent of the noncommutative matrices order was carried out in [11,12], it has the form

$$C = [(AB^{-1})^{1/2}B]^2 = [(BA^{-1})^{1/2}A]^2. \tag{3.12}$$

This operation is valid for the nonsingular matrices of arbitrary order. If the matrices A and B belong to the Abelian or non-Abelian group g then also* $C \in g$.

If matrices A and B belong to the groups $SL(2,R)$ or $SU(1,1)$ then (3.12) has the descriptive geometric representation. This matrices in this case may be presented as the geodesic vectors originating from the common point on the Lobachevsky plane in its representation on the Poincare map. The matrix C corresponds to the diagonal of "parallelogram" originating from the same point [11,12]. If the matrices in (3.12) near to the identity ones and they may be expanded into series then (3.12) goes over to the vector addition on the Euclidean plane. The second diagonal of "parallelogam" corresponds to the antisymmetric operation. It should be noted that geometric interpretation of propagators allows one to determine the path product integral measure and to formulate its variational principle. The measure is the square element of the Lobachevsky plane $dudv/(4u^2)$, the variational principle has the form $\delta(TrQ) = 0$ [7,11].

The circumstance that the propagator composition may be mapped as the geodesic vectors addition on the non-Euclidean plane is not striking as far as the propagator logarithms space even for unidimensional Schrodinger equation, as it was shown above, is the Lobachevsky plane.

4. Comparison of superposition and composition principles

The differences between two principles allowing to take into account the contribution of alternative paths into the complete propagator appear expressively in the simplest problems containing equal noncommutative propagators. These requirements are fulfilled in the "mental experiment" with two infinitesimal slits situated at the two-media boundary. We shall use this experiment for the comparison of observables, calculated in accordance with two different principles, with respect to its transformation properties and conservation laws fulfilment.

Let A and B are two different propagators corresponding to two different paths from the initial point S to the final point R defined by the geometry of problem, Figure 1. Let us suppose that the conditions (2.2) and corresponding conditions for spinor defined at the initial point are the same for both paths. In accordance with (3.7) the propagators A and B have the form [7,16]

$$\begin{aligned} A = e^{\mathbf{a}\sigma} = e^{iM_1\sigma_3}e^{L\sigma_1}e^{iN_1\sigma_3}, \\ B = e^{\mathbf{b}\sigma} = e^{iM_2\sigma_3}e^{L\sigma_1}e^{iN_2\sigma_3}, \end{aligned} \tag{4.1}$$

where $N_1 = k_1 s_A, N_2 = k_1 s_B, M_1 = k_2 r_A, M_2 = k_2 r_B, L = \frac{1}{2}\ln\frac{k_2}{k_1}$, k_1 and k_2 are indexes of refraction in different media, $s_{A,B}$ and $r_{A,B}$ are the path lengths in media with k_1 and k_2 respectively.

*The antisymmetric binary operation conserving the group belonging has the form $D = (AB)^{1/2}B^{-1}$ where $(BA)^{1/2}A^{-1} = D^{-1}$ and if $A, B \in g$ then also $D \in g$.

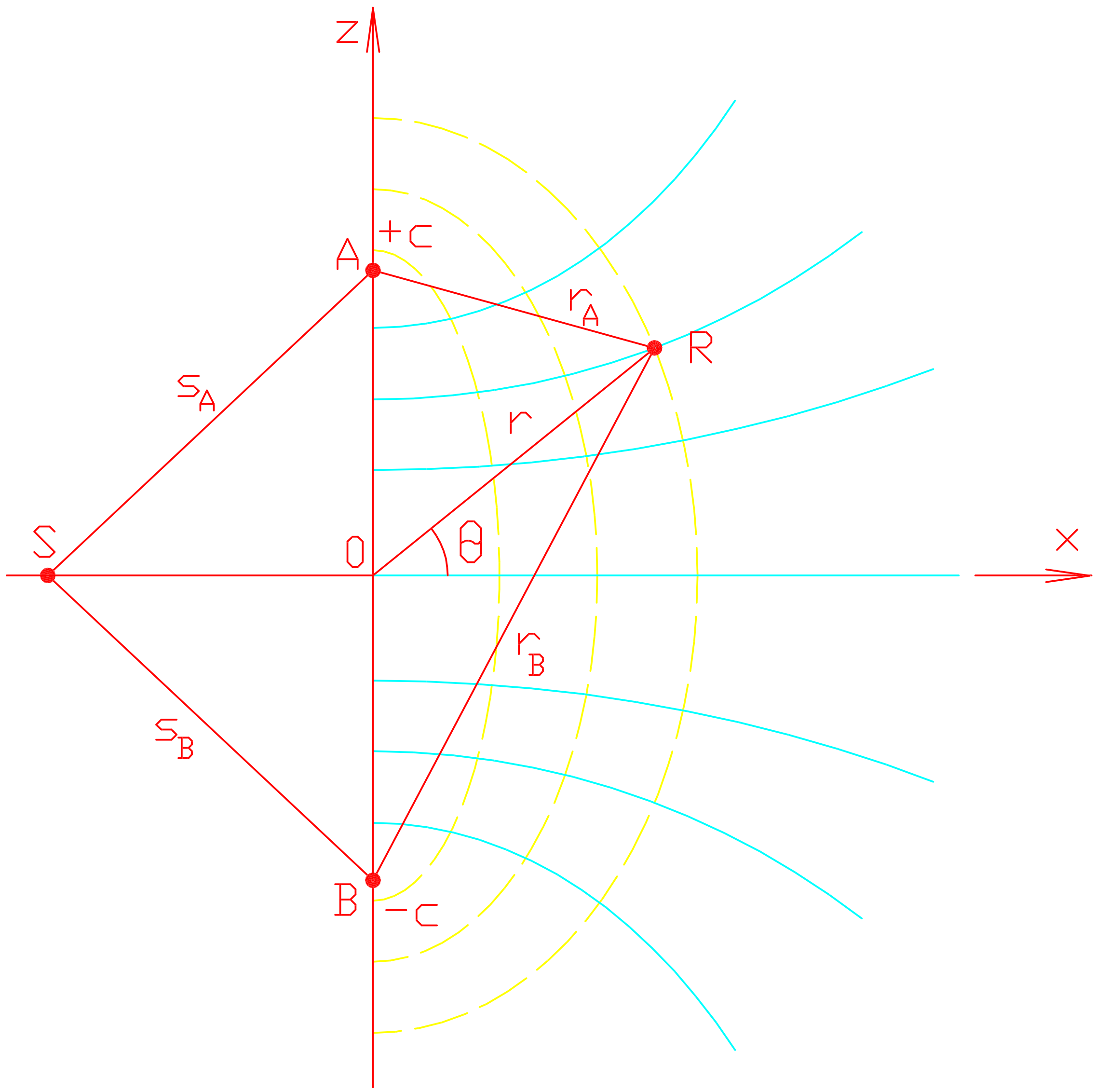

Figure 1: The experiment with two slits situated at the boundary of two media.

Let us calculate the hyperbolic first order moment [7] Q of two propagators A and B on the Lobachevsky plane excluding power 2 in (3.12)

$$Q = (e^{\mathbf{a}\sigma} e^{-\mathbf{b}\sigma})^{1/2} e^{\mathbf{b}\sigma} = (e^{\mathbf{b}\sigma} e^{-\mathbf{a}\sigma})^{1/2} e^{\mathbf{a}\sigma}. \tag{4.2}$$

One has a single slit propagator $Q = \exp(\mathbf{a}\sigma)$ under the coincidence of both slits: $\mathbf{b} \to \mathbf{a}$. This case may be considered as the shut-down of one slit.

The obvious form of the complete propagator (4.2) including both paths in the case of two slits may be expressed as

$$Q = \frac{\exp(\mathbf{a}\sigma) + \exp(\mathbf{b}\sigma)}{\sqrt{(\cosh a + \cosh b)^2 - (\mathbf{n}_a \sinh a + \mathbf{n}_b \sinh b)^2}}\,, \tag{4.3}$$

where the expression under radical sign is $\det[\exp(\mathbf{a}\sigma) + \exp(\mathbf{b}\sigma)]$ therefore $\det Q = 1$.

It follows from expressions (4.3) and (4.1)

$$Q = (\cosh^2 L \cos^2 \frac{(N_1 - N_2) + (M_1 - M_2)}{2} - \sinh^2 L \cos^2 \frac{(N_1 - N_2) - (M_1 - M_2)}{2})^{-1/2} \times$$

$$\times \begin{Vmatrix} \cosh L \cos \frac{(N_1-N_2)+(M_1-M_2)}{2} \times & \sinh L \cos \frac{(N_1-N_2)-(M_1-M_2)}{2} \times \\ \times \exp(i \frac{(N_1+N_2)+(M_1+M_2)}{2}) & \times \exp(-i \frac{(N_1+N_2)-(M_1+M_2)}{2}) \\ & \\ \sinh L \cos \frac{(N_1-N_2)-(M_1-M_2)}{2} \times & \cosh L \cos \frac{(N_1-N_2)+(M_1-M_2)}{2} \times \\ \times \exp(i \frac{(N_1+N_2)-(M_1+M_2)}{2}) & \times \exp(-i \frac{(N_1+N_2)+(M_1+M_2)}{2}) \end{Vmatrix} =$$

$$= \frac{1}{2} \cdot \frac{e^{iM_1\sigma_3} e^{L\sigma_1} e^{iN_1\sigma_3} + e^{iM_2\sigma_3} e^{L\sigma_1} e^{iN_2\sigma_3}}{\sqrt{\cos^2 \frac{(N_1-N_2)+(M_1-M_2)}{2} - \sinh^2 L \sin(N_1 - N_2) \sin(M_1 - M_2)}}. \quad (4.4)$$

It is obviously that $Q \in SU(1,1)$.

The spinor components at the final point R may be expressed by means of matrix Q from (4.4) in the expression $\Phi(R) = Q\Phi(S)$ therefore all observables $j_s(R,Q)$ may be calculated at the final point R.

The direct calculations or the fact that $Q \in SU(1,1)$ leads to the condition $j_3(R,Q)= j_3(S)$, i.e. the current j_3 is conserved.

Accordingly to the composition principle (4.2) one has

$$j_0(R,Q) = (a^2 - b^2)^{-1}[(a^2 + b^2)j_0(S) + 2ab\cos(\alpha + \beta)j_1(S) + 2ab\sin(\alpha + \beta)j_2(S)], \quad (4.5)$$

$$\begin{aligned} j_1(R,Q) = \quad & (a^2 - b^2)^{-1}[2ab\cos(\alpha - \beta)j_0(S) + (a^2\cos 2\alpha + b^2\cos 2\beta)j_1(S) + \\ & +(a^2\sin 2\alpha + b^2\sin 2\beta)j_2(S)], \end{aligned} \quad (4.6)$$

$$\begin{aligned} j_2(R,Q) = \quad & (a^2 - b^2)^{-1}[-2ab\sin(\alpha - \beta)j_0(S) - (a^2\sin 2\alpha - b^2\sin 2\beta)j_1(S) + \\ & +(a^2\cos 2\alpha - b^2\cos 2\beta)j_2(S)], \end{aligned} \quad (4.7)$$

$$j_3(R,Q) = j_3(S), \quad (4.8)$$

where

$$\begin{aligned} a = \cosh(\tfrac{1}{2}\ln\tfrac{k_2}{k_1}) \cdot \cos\tfrac{(N_1-N_2)+(M_1-M_2)}{2}, \quad & \alpha + \beta = N_1 + N_2, \\ b = \sinh(\tfrac{1}{2}\ln\tfrac{k_2}{k_1}) \cdot \cos\tfrac{(N_1-N_2)-(M_1-M_2)}{2}, \quad & \alpha - \beta = M_1 + M_2. \end{aligned}$$

It follows from (4.5) - (4.8) that $j_s(R,Q)$ with $s = 0,1,2$ do not consist of $j_3(S)$. Besides, variables $j_0(S)$ и $j_1(S)$ combine with even functions of coordinates and $j_2(S)$ combines with odd functions of coordinates in the expressions for scalar observables $j_0(R,Q)$ and $j_1(R,Q)$ then both last variables do not change the sign under coordinate system inversion. At the same time the variables $j_0(S)$ and $j_1(S)$ combine with odd functions of coordinates and variable $j_2(S)$ combines with even functions of coordinates in the expression for vector observable $j_2(R,Q)$ then last variable changes its sign under the same procedure.

Thus the first order hyperbolic moment calculated accordingly to (4.2) leads to the observables having correct transformation properties under coordinate system inversion and satisfying to the necessary conservation laws (2.4) and (2.15). At the same time three independent observables defined at the initial point allow one to calculate three observables at the final point.

In accordance with the superposition principle one has $\Phi(R) = (A+B)\Phi(S)$ with A and B from (4.1). In the case of two slits coincidence, there are $\mathbf{b} \to \mathbf{a}$ and $B \to A$, but $(A+B) \not\to A$ at the same time. If one changes $(A+B) \to (A+B)/2$ then the complete propagator goes to A therefore we shall accept that the complete two slits propagator has the following form

$$P = (e^{iM_1\sigma_3}e^{L\sigma_1}e^{iN_1\sigma_3} + e^{iM_2\sigma_3}e^{L\sigma_1}e^{iN_2\sigma_3})/2. \tag{4.9}$$

The expression (4.9) corresponding to the superposition principle differs from the expression (4.4) corresponding to the composition principle on the Lobachevsky plane with the radical in the denominator of last one. Therefore all observables $j_s(R,Q)$ calculated at the final point R by means of propagator Q from (4.4) differ of the observables $j_s(R,P)$ calculated at the same point by means of propagator P from (4.9) with the common factor

$$\begin{aligned} j_s(R,P) &= [\cos^2 \tfrac{(N_1-N_2)+(M_1-M_2)}{2} - \\ &- \sinh^2 L \sin{(N_1-N_2)} \sin{(M_1-M_2)}] \cdot j_s(R,Q). \end{aligned} \tag{4.10}$$

This factor is the determinant of two propagators sum, it depends on the medium properties along these two paths.

As far as $j_3(R,Q) = j_3(S)$ then the expressions $j_3(R,P)$ and $j_3(S)$ differ with the same factor. It means that $j_3 \neq const$ therefore the superposition principle leads to the violation of the necessary conservation law.

The "probability density" ρ calculated accordingly to the superposition and composition principles have the same distinction due to the expression $\rho = (j_0 + j_1)/(2k)$.

An arbitrary matrix from the group $SU(1,1)$ may be expressed in the form (4.1) therefore the expression (4.4) and corresponding expressions for observables (4.5) – (4.8) may be used for the analysis of propagators composition results for any two paths where the potential has an arbitrary coordinate dependence. However the parameters N, L and M will not have such simple interpretation as in (4.1) in this case.

In the case when so the initial and the final points as two coinciding slits are situated on the straight line which is perpendicular to the boundary the expressions (4.4) – (4.8) lead to the same expressions for propagator, reflection and transition coefficients as in the one-dimensional case calculations [11,14]. The sum of these coefficients satisfies to the condition $R + T = 1$ due to $Q \in SU(1,1)$ at the same time.

Let us take identical media, $L = 0$, then the expressions (4.5) – (4.8) have the form

$$j_0(R,Q) = j_0(S), j_3(R,Q) = j_3(S), \tag{4.11}$$

$$j_1(R,Q) = \cos 2\alpha \cdot j_1(S) + \sin 2\alpha \cdot j_2(S), \tag{4.12}$$

$$j_2(R,Q) = -\sin 2\alpha \cdot j_1(S) + \cos 2\alpha \cdot j_2(S), \tag{4.13}$$

where $2\alpha = N_1 + M_1 + N_2 + M_2$.

If the positions of initial point S and both slits are fixed (N_1 and N_2 are constant) then the surfaces of constant probability density $\rho = (j_0 + j_1)/(2k)$ are ellipsoids of revolution $M_1 + M_2 = const$ with slits at their foci.

The diffusion current of probability density $\nabla\rho = \nabla(j_0 + j_1)/(2k) = j_2 \cdot \nabla(r_B + r_A)$ is orthogonal to the surfaces $\rho = const$, its value oscillates in the opposite phase to ρ in three-dimensional space .

Let the screen is situated in the plane $x = 0$ and two slits are situated at the $z = \pm c$ on the z-axis, figure 1. Restricting with the case of point S on the x-axis and considering the contour $SARBS$ formed by two paths as loop trajectory one may obtain, taking into account the quantum conditions for loops [7,16] including signs for N_i, M_i in (4.1), $M_1 - M_2 = k(r_B - r_A) = \pi n$. This condition is equivalent to

$$r_B - r_A = n(\lambda/2), n = 0, \pm 1, ..., \tag{4.14}$$

where $k = 2\pi/\lambda$, λ is wavelength. In whole space at the right side of the screen one has $-2c \leq r_B - r_A \leq 2c$. The maximal value of n is $n_{max} = [2c/(\lambda/2)]$ and on the plane $z = 0$ ($r_B = r_A$) $n = 0$ at the same time. Therefore the expression (4.14) defines the restricted discrete set of hyperboloids of revolution which are confocal to the ellipsoids mentioned above.

The current directions are tangent to the hyperbolas which are the conic sections of hyperboloids by planes passing through the x-axis.

Using the identity $r_B - r_A = 4rc\sin\theta/(r_B + r_A)$, figure 1, one may obtain the condition $r_B - r_A = 2c\sin\theta = n\lambda/2$ from (4.14) for large r. This condition defines the positions of maxima and minima for fringes beyond the screen with two slits.

It should be mentioned that maxima of interference fringes correspondent to extrema of "probability density" diffusion current $\nabla\rho$, but not to ρ. The expression in square brackets in (4.10) is equal to unit at these points, therefore the current $\mathbf{j}_3$ is conserved there both in the case of use the ordinary and the Lobachevsky superposition principles. In the case of use the Lobachevsky one this current is conserved everywhere.

The analysis of conservation laws in their differential form carried out in [16] leads to the same conclusion as it obtained in algebraic form [7] in section 4 of the present paper.

5. Conclusion

The fulfilment of conservation laws and their relation with the group-theoretic requirements for propagators in the stationary problems of quantum mechanics described with the partial Schrodinger equation with real potential were investigated in this paper. It was shown that the nonrelativistic quantum mechanics is not the consecutive group-theoretic theory. In accordance with the Noether theorems it may lead to violation of the conservation laws.

The complete set of observables for the Schrodinger equation was obtained, only three of them are independent. The transformation properties of complete propagator for spinor were obtained on the base of transormation properties of the observables and conservation laws for them. The complete propagator is constructed in the set of partial propagators, both of them belong to the same group. The complete propagator is the path product integral.

The connection of complete propagator for the Schrodinger equation with the Lobachevsky geometry allowed one to establish the non-Euclidean superposition principle on the hyperbolic plane obtained on the base only group-theoretic requirements to the theory.

This principle allows one to develop the consecutive group-theoretic quantum mechanics, to obtain the measure for the path product integral and to formulate the variational principle. The Lobachevsky superposition principle may be considered as the generalization of the ordinary Euclidean superposition principle, it turns to the last one in the correspondent range of parameters.

The results of calculations for observables obtained in accordance with the ordinary superposition principle and in accordance with the Lobachevsky one are compared for the "mental experiment" with two slits situated at the boundary of two media. It was shown that the use of ordinary superposition principle leads to the violation of conservation laws under some conditions. The use of the Lobachevsky superposition principle leads to fulfilment of the conservation laws everywhere at the same time.

One of the consequence of conservation laws violation under use of the Euclidean superposition principle in the nonrelativistic quantum mechanics due to violation of the group-theoretic structure of the theory shown above is following.

The logical completeness of the quantum electrodynamics is absent now. The testimony of this fact is the presence of divergencies in the observables calculations there. It may be excluded only by means of renormalization technique which does not enter the theory first principles. One of the possible cause may be violation of some conservation laws in QED which has to be a consequence of the violation group-theoretic requirements to the theory in accordance with the Noether theorems.

QED is the combination of two fundamental theories: the special relativity which is the group-theoretic theory and the nonrelativistic quantum mechanics which is not the consecutive group-theoretic theory as it was mentioned above. It is clear that the combination of two such theories can not lead to the group-theoretic QED. According to the Noether theorems this circumstance may lead to the violation of some conservation laws. Therefore the violation of conservation laws existing also as it was shown above in the nonrelativistic quantum mechanics may be considered as an analog of divergencies in QED.

The successive group-theoretic approach of the nonrelativistic quantum mechanics developed on the base of path product integral leading to the Lobachevsky superposition principle leads to fulfilment of the conservation laws everywhere. It allows one to wait that the development of QED without divergencies may be carried out also on the same base under fulfilment of the special relativity groups requirements both for partial propagators and for their composition into complete one at the same time.

Acknowledgements

I would like to thank S.T.Beliaev, A.V.Gaponov, S.S.Gershtein, V.I.Kogan and A.G.Litvak for attention and support of this work. The technical assistance of Mrs. L.A.Lunina is greatly appreciated.

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